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TYPE $1,1$-OPERATORS ON SPACES OF TEMPERATE DISTRIBUTIONS

JON JOHNSEN

ABSTRACT. This paper is a follow-up on the author's general definition of pseudo-differential operators of type $1,1$, in Hörmander's sense. It is shown that such operators are always defined on the smooth functions that are temperate; and moreover are defined and continuous on the space of temperate distributions, whenever they fulfil the twisted diagonal condition of Hörmander, or more generally when they belong to the self-adjoint subclass. Continuity in L_p-Sobolev spaces and Hölder–Zygmund spaces, and more generally in Besov and Lizorkin–Triebel spaces, is for positive smoothness also proved on the basis of the definition. These continuity results are extended to arbitrary real smoothness indices for operators that fulfil the twisted diagonal condition or belong to the self-adjoint subclass. With systematic Littlewood–Paley analysis the well-known paradifferential decomposition is also derived for type $1,1$-operators. The proofs are based on a spectral support rule for pseudo-differential operators in combination with pointwise estimates in terms of maximal functions.

1. INTRODUCTION

1.1. **Background.** Pseudo-differential operators of type $1,1$ have almost from the outset been shown to have rather special properties, due to initial investigations in 1972 in the thesis of Ching [Chi72] and unpublished lecture notes of Stein (cf [Ste93]); and again in 1978 by Parenti and Rodino [PR78].

A more substantial understanding of their theory and applications was obtained in the following decade through works of Meyer [Mey81a, Mey81b], Bony [Bon81], Bourdaud [Bou82, Bou83, Bou88b, Bou88a], Hörmander [Hör88, Hör89]; cf also the exposition in [Hör97, Ch. 9]. In recent years progress in the subject has been made by the author, with [Joh04, Joh05] devoted to the L_p-theory and the fact that Lizorkin–Triebel spaces $F^s_{p,q}$ are optimal for certain borderlines.

However, the first formal definition of general type $1,1$-operators was given by the author in [Joh08b] as the basis for a discussion of unclosability, hypoellipticity, non-preservation of wavefront sets and spectral support rules. The present paper continues the work in [Joh08b] with a much deeper study of type $1,1$-operators on $\mathscr{S}'(\mathbb{R}^n)$ and its subspaces.

By definition, the symbol $a(x,\eta)$ of a type $1,1$-operator of order $d\in\mathbb{R}$ fulfils

$$|D^\alpha_\eta D^\beta_x a(x,\eta)|\le C_{\alpha,\beta}(1+|\eta|)^{d-|\alpha|+|\beta|}\quad\text{for}\quad x,\eta\in\mathbb{R}^n. \tag{1.1}$$

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The corresponding operator is $a(x,D)u = (2\pi)^{-n}\int e^{-\mathrm{i}x\cdot\eta}a(x,\eta)\hat{u}(\eta)\,d\eta$ if u is a Schwartz function, ie $u\in\mathscr{S}(\mathbb{R}^n)$. But for $u\in\mathscr{S}'\setminus\mathscr{S}$ it is a question to settle whether u belongs to the domain or not; for this purpose a general definition was presented in [Joh08b], cf (1.8) below.

The pathologies of type $1,1$-operators are without doubt reflecting the fact that, most interestingly, this operator class has important applications to non-linear problems.

This was first described around 1980 by Meyer [Mey81a, Mey81b], who discovered that a composition operator $u\mapsto F\circ u = F(u)$ with $F\in C^\infty$, $F(0)=0$, can be decomposed when acting on $u\in\bigcup_{s>n/p}H^s_p(\mathbb{R}^n)$ by means of a specific u-dependent symbol $a_u(x,\eta)\in S^0_{1,1}$ as

$$F(u(x)) = a_u(x,D)u(x). \tag{1.2}$$

He also showed that $a_u(x,D)$ is bounded on H^t_r for $t>0$, so the fact that the non-linear map $F(u)$ sends H^s_p into itself results at once from (1.2) for $t=s$ and $r=p$ — indeed, this celebrated proof is particularly elegant for non-integer $s>n/p$.

Secondly, it became clear at the same time that type $1,1$-operators enter the paradifferential calculus of Bony [Bon81] and the microlocal inversion for nonlinear partial differential equations of the form

$$G(x,(D^\alpha_x u(x))_{|\alpha|\le m}) = 0. \tag{1.3}$$

This was explicated eg by Hörmander, who devoted Chapter 10 of [Hör97] to this subject. The resulting framework was used eg by Hérau [Hér02] in a study of hypoellipticity of (1.3).

Thirdly, type $1,1$-operators were recently used by the author in the analysis of semi-linear boundary problems [Joh08a]. Because of the novelty, this will now be sketched through a typical example: in a bounded C^∞-region $\Omega\subset\mathbb{R}^n$ (with normal derivatives $\gamma_j u = (\vec{n}\cdot\nabla)^j u$ at the boundary $\partial\Omega$, $\Delta = \partial^2_{x_1}+\dots+\partial^2_{x_n}$), let $u(x)$ solve the perturbed ℓ-harmonic Dirichlét problem

$$(-\Delta)^\ell u + u^2 = f \quad\text{in}\quad \Omega, \qquad \gamma_j u = \varphi_j \quad\text{on}\quad \partial\Omega,\ j=0,\dots,\ell-1. \tag{1.4}$$

For such problems the parametrix construction of [Joh08a] yields the solution formula

$$u = P^{(N)}_u(R_\ell f + K_0\varphi_0 + \dots + K_{\ell-1}\varphi_{\ell-1}) + (R_\ell L_u)^N u, \tag{1.5}$$

where the parametrix $P^{(N)}_u$ is the linear map

$$P^{(N)}_u = I + R_\ell L_u + \dots + (R_\ell L_u)^{N-1} \tag{1.6}$$

in which the exact paralinearisation L_u of u^2 is a main ingredient, with the sign convention $-L_u(u) = u^2$. (R_ℓ, $K_0,\dots,K_{\ell-1}$ resolve the linear problem, cf the case $L_u\equiv 0$ in (1.5).)

Formula (1.5) shows directly that the regularity of u will be uninfluenced by the non-linear term u^2: the parametrix $P^{(N)}_u$ is of order 0 for every N, while the remainder $(R_\ell L_u)^N u$ will be in $C^k(\overline{\Omega})$ for every fixed k if N is taken large enough (in both cases because $R_\ell L_u$ will have negative order if the given u has a certain weak a priori regularity). These inferences may be justified using parameter domains as in [Joh08a].

Moreover, in subregions $\Xi\Subset\Omega$, extra regularity properties of f carry over to u (eg, if $f|_\Xi$ is C^∞ so is $u|_\Xi$). This also follows from (1.5), because L_u factors through a type $1,1$-operator A_u; ie when r_Ω and ℓ_Ω denote restriction to and a linear extension from Ω,

$$L_u = r_\Omega A_u \ell_\Omega, \qquad A_u\in \mathrm{OP}(S^\infty_{1,1}). \tag{1.7}$$

Hence, by inserting cut-off functions supported in Ξ into (1.5) in a well-known way, cf [Joh08a, Thm. 7.8], the *pseudo-local* property of A_u in (1.7) leads to improved regularity of u locally in Ξ, to the extent permitted by the data f.

However, the pseudo-local property of general type $1,1$-operators was first proved recently by the author in [Joh08b]. It was anticipated more than three decades ago by Parenti and Rodino [PR78], who gave an inspiring but incomplete indication, as they did not assign a specific meaning to $a(x,D)u$ for $u \in \mathscr{S}' \setminus C_0^\infty$.

A rigorous definition of type $1,1$ operators was first given in [Joh08b], taking into account that in some cases they can only be defined on proper subspaces $E \subset \mathscr{S}'(\mathbb{R}^n)$. Indeed, it was proposed in [Joh08b] to stipulate that $u \in D(a(x,D))$ and to set

$$a(x,D)u := \lim_{m\to\infty} \operatorname{OP}(\psi(2^{-m}D_x)a(x,\eta)\psi(2^{-m}\eta))u \tag{1.8}$$

whenever this limit exists in $\mathscr{D}'(\mathbb{R}^n)$ for all the $\psi \in C_0^\infty(\mathbb{R}^n)$ with $\psi = 1$ in a neighbourhood of the origin and does not depend on such ψ.

This unconventional definition, by *vanishing frequency modulation*, could be seen as a rewriting of the usual one, which is suitable for the present general symbols. (Clearly (1.8) gives back the integral after (1.1) if $u \in \mathscr{S}$; in case $a \in S^d_{1,0}$ this identification extends to $u \in \mathscr{S}'$ by duality.) Formally it is reminiscent of oscillatory integrals, now with the proviso that $u \in D(a(x,D))$ when the regularisation yields a limit independent of the integration factor.

Of course the frequencies of $a(\cdot,\eta)$ are not modified using an integration factor in the strict sense here, but rather with the Fourier multiplier $\psi(2^{-m}D_x)$. This difference is emphasized because the use of $\psi(2^{-m}D_x)$ gives easy access to Littlewood–Paley analysis of $a(x,D)$.

The definition was also investigated in [Joh08b] from several other perspectives. Some of these will be recalled further below, but briefly mentioned, (1.8) was proved to be maximal among the definitions of $A = a(x,D)$ that is both compatible with $\operatorname{OP}(S^{-\infty})$ and stable under the limit in (1.8); eg A is always defined on $\mathscr{F}^{-1}\mathscr{E}'$, it is pseudo-local but does change wavefront sets in certain cases; and A transports supports via the distribution kernel, ie $\operatorname{supp} Au \subset \operatorname{supp} K \circ \operatorname{supp} u$ when $u \in D(A)$ has compact support, with a similar *spectral* support rule for $\operatorname{supp} \hat{u}$ recalled Appendix B below (including general versions without compactness assumptions); cf (1.18).

For the Weyl calculus, Hörmander [Hör88] noted that type $1,1$-operators do not fit well, as Ching's operator can have discontinuous Weyl-symbol. Boulkhemair [Bou95, Bou99] showed that insertion of $a(x,\eta)$ in $S^d_{1,1}$ into the Weyl operator $\iint e^{\mathrm{i}(x-y)\cdot\eta} a(\frac{x+y}{2},\eta)u(y)\,dy\,d\eta/(2\pi)^n$ may give peculiar properties. Eg, already for Ching's symbol with $d = 0$, the real or imaginary part gives a Weyl operator that is unbounded on H^s for *every* $s \in \mathbb{R}$.

For more remarks on the historic development of the subject the reader may refer to Section 2.2 below. A more thorough presentation was given in the introduction of [Joh08b].

1.2. **Review of present results.** The purpose of this paper is to continue the general study in [Joh08b] and support the definition in (1.8) with further consequences.

First of all this means to address the hitherto untreated question: under which conditions is a given type $1,1$-operator $a(x,D)$ an everywhere defined and continuous map

$$a(x,D)\colon \mathscr{S}'(\mathbb{R}^n) \to \mathscr{S}'(\mathbb{R}^n) \quad ? \tag{1.9}$$

For this it is shown here to be sufficient that $a(x,\eta)$ fulfils Hörmander's twisted diagonal condition, that is, the partially Fourier transformed symbol $\hat{a}(\xi,\eta) = \mathscr{F}_{x\to\xi}a(x,\eta)$ should vanish in a conical neighbourhood of a non-compact part of the twisted diagonal given by $\xi+\eta=0$ in $\mathbb{R}^n\times\mathbb{R}^n$; or more precisely, for some $B\geq 1$

$$\hat{a}(\xi,\eta) = 0 \quad \text{when} \quad B(|\xi+\eta|+1) < |\eta|. \tag{1.10}$$

It should perhaps be noted that $\hat{a}(\xi,\eta)$ is a natural object to consider, as it is related (cf [Joh08b, Prop. 4.2]) both to the kernel K of $a(x,D)$ and to the kernel $\mathscr{K}$ of $\mathscr{F}^{-1}a(x,D)\mathscr{F}$,

$$\mathscr{K}(\xi,\eta) = (2\pi)^{-n}\hat{a}(\xi-\eta,\eta) = (2\pi)^{-n}\mathscr{F}_{(x,y)\to(\xi,\eta)}K(x,-y). \tag{1.11}$$

More generally than (1.10), (1.9) is proved for the $a(x,\eta)$ in $S^d_{1,1}$ that just satisfy Hörmander's twisted diagonal condition of order σ for all $\sigma\in\mathbb{R}$. This means that for some $c_{\alpha,\sigma}$,

$$\sup_{x\in\mathbb{R}^n,\,R>0} R^{|\alpha|-d}\Big(\int_{R\leq|\eta|\leq 2R} |D^\alpha_\eta a_{\chi,\varepsilon}(x,\eta)|^2\,\frac{d\eta}{R^n}\Big)^{1/2} \leq c_{\alpha,\sigma}\varepsilon^{\sigma+n/2-|\alpha|} \quad \text{for} \quad 0<\varepsilon<1. \tag{1.12}$$

In this asymptotic formula $\hat{a}_{\chi,\varepsilon}$ denotes a specific localisation of $\hat{a}(x,\eta)$ to the conical neighbourhood $|\xi+\eta|+1\leq 2\varepsilon|\eta|$ of the twisted diagonal. The details behind this are given in Section 2.2, where also the consequences of (1.10), (1.12) for Sobolev space continuity is recalled.

These two sufficient conditions for (1.9) should be completely new in the sense that the question has, seemingly, been neither raised nor treated before.

It is also shown that every $a(x,D)$ of type $1,1$ is defined on the maximal space of smooth functions $C^\infty\bigcap\mathscr{S}'$. More precisely, it restricts to a map

$$a(x,D)\colon C^\infty(\mathbb{R}^n)\bigcap\mathscr{S}'(\mathbb{R}^n)\to C^\infty(\mathbb{R}^n). \tag{1.13}$$

This relies on and improves an extension of Bourdaud [Bou88a] to the space $\mathscr{O}_M(\mathbb{R}^n)$ of slowly increasing smooth functions. Since the map in (1.13) leaves $\mathscr{O}_M$ invariant, it also completes the earlier result $a(x,D)\colon \mathscr{S}+\mathscr{F}^{-1}\mathscr{E}'\to\mathscr{O}_M$ of the author [Joh05, Joh08b].

The usefulness of the definition (1.8) is more substantial than this, for it furthermore allows Littlewood–Paley analysis via the well-known paradifferential splitting with dyadic coronas (as used by Bony [Bon81], details are given in Section 5),

$$a(x,D) = a^{(1)}_\psi(x,D) + a^{(2)}_\psi(x,D) + a^{(3)}_\psi(x,D). \tag{1.14}$$

This decomposition follows directly from the bilinearity with respect to ψ in definition (1.8), as was briefly mentioned in [Joh08b, Sect. 9]. But as accounted for here, all terms on the right-hand side are also in $\mathrm{OP}(S^d_{1,1})$ when $a(x,D)$ fulfils (1.10) or (1.12) for all $\sigma\in\mathbb{R}$.

Since the 1980's splittings like (1.14) have been used in numerous proofs of continuity in Sobolev spaces H^s_p and Hölder–Zygmund spaces C^s_*, or Besov and Lizorkin–Triebel scales $B^s_{p,q}$ and $F^s_{p,q}$. For type $1,1$ operators such techniques have been used by the author in [Joh04, Joh05, Joh08b] and earlier by Bourdaud [Bou82, Bou83, Bou88a], Marschall [Mar91], Runst [Run85].

These works are followed up here with the first full proof (based on (1.8)) that every type $1,1$-operator $a(x,D)$ is bounded for all $s>0$, $1<p<\infty$

$$a(x,D)\colon H^{s+d}_p(\mathbb{R}^n)\to H^s_p(\mathbb{R}^n), \qquad a(x,D)\colon C^{s+d}_*(\mathbb{R}^n)\to C^s_*(\mathbb{R}^n); \tag{1.15}$$

and it is proved that this extends to every $s\in\mathbb{R}$ when the twisted diagonal condition of order σ in (1.12) holds for all $\sigma\in\mathbb{R}$. This gives a generalisation to the L_p-setting of a result of Hörmander [Hör88, Hör89], who showed the extendability to $s\le 0$ for $p=2$ under the conditions (1.10) or (1.12) for all σ.

The results in (1.15) are actually shown here as corollaries of similar results for the general $B^s_{p,q}$ and $F^s_{p,q}$ scales, including the extension to $s\le 0$ when (1.12) holds for all σ. These L_p-results exploit both (1.14) and the discussion of $\mathscr{S}'$-continuity after (1.9) in a natural way (large parts of the proofs are the same), hence should be well motivated in this article.

However, the decomposition (1.14) is also interesting because it is a main source of operators with the property (1.9). Indeed, both $a^{(1)}_\psi(x,D)$ and $a^{(3)}_\psi(x,D)$ always satisfy the twisted diagonal condition in (1.10), hence are harmless in the sense that they are defined for all $u\in\mathscr{S}'$ by (1.9).

Therefore it is the 'symmetric' term $a^{(2)}_\psi(x,D)$ which may cause $a(x,D)u$ to be undefined, as was previously known eg for $u\in\bigcup_s H^s$; cf [Joh05]. More precisely, the infinite series defining $a^{(2)}_\psi(x,D)u$ need not converge for all $u\in\mathscr{S}'$, but it is shown here to do so whenever $a(x,\eta)$ fulfils the twisted diagonal condition of order σ for every $\sigma\in\mathbb{R}$.

In comparison convergence of the series for $a^{(1)}_\psi(x,D)u$ and $a^{(3)}_\psi(x,D)u$ is verified below for all $u\in\mathscr{S}'$, $a\in S^\infty_{1,1}$. Thereby both the splitting (1.14) itself and the convenient infinite series expressions have been carried over to the framework of type $1,1$-operators.

Although the convergence results are hardly surprising, they rely on two techniques introduced recently in works of the author. One is a *pointwise* estimate

$$|a(x,D)u(x)|\le c\,u^*(x),\qquad x\in\mathbb{R}^n, \tag{1.16}$$

cf Section 3, in terms of the Peetre–Fefferman–Stein maximal function

$$u^*(x)=\frac{|u(x-y)|}{(1+R|y|)^N},\quad \text{when}\quad \operatorname{supp}\hat{u}\subset\overline{B}(0,R). \tag{1.17}$$

The other ingredient is a *spectral support rule*, that controls $\operatorname{supp}\mathscr{F}(a(x,D)u)$ in terms of the supports of $\hat{u}$ and of $\mathscr{K}(\xi,\eta)$; (1.11). Eg in case $\operatorname{supp}\hat{u}$ is compact,

$$\operatorname{supp}\mathscr{F}(a(x,D)u)\subset\operatorname{supp}\mathscr{K}\circ\operatorname{supp}\hat{u}=\{\,\xi+\eta\mid(\xi,\eta)\in\operatorname{supp}\hat{a},\ \eta\in\operatorname{supp}\hat{u}\,\}. \tag{1.18}$$

This was proved in [Joh04, Joh05] with a more general version in [Joh08b]. The purpose is to avoid elementary symbols, that were introduced by Coifman and Meyer [CM78] because $\operatorname{supp}\mathscr{F}(a(x,D)u)$ is easy to control for these. Indeed, they are symbols given in the form $a(x,\eta)=\sum m_j(x)\Phi_j(\eta)$ for a sequence (m_j) in L_∞ and a Littlewood–Paley partition of unity $1=\sum\Phi_j$, whence $\mathscr{F}a(x,D)u=(2\pi)^{-n}\sum\hat{m}_j*(\Phi_j\hat{u})$ is a finite sum for which the support rule for convolutions yields a proof of (1.18) in this case. A review of (1.18) is given in Appendix B, including an equally easy proof for arbitrary $a\in S^d_{1,0}$.

However, elementary symbols are not just technically redundant because of (1.18), they would also be particularly cumbersome to use in the context of type $1,1$-symbols, as (1.8) would lead to a double-limit procedure. So in the proof of (1.14) and the L_p-theory based on it, (1.18) yields a significant simplification.

Remark 1.1. The spectral support rule (1.18) shows clearly that the role of the twisted diagonal condition (1.10) is to ensure that $a(x,D)$ cannot change (large) frequencies in $\operatorname{supp}\hat{u}$ to 0. In fact, (1.10) means that ξ cannot be close to $-\eta$ when $(\xi,\eta)\in\operatorname{supp}\hat{a}$, which by (1.18) means that $\eta\in\operatorname{supp}\hat{u}$ will be changed to the frequency $\xi+\eta\neq 0$.

Notation is settled in Section 2 along with basics on operators of type $1,1$. In Section 3 the pointwise estimates are recalled from [Joh10a], and extended to a version for frequency modulated operators. Section 4 gives a precise analysis of the self-adjoint part of $S^d_{1,1}$, relying on the results and methods from Hörmander's lecture notes [Hör97, Ch. 9]; with consequences derived from the present operator definition. Littlewood–Paley analysis of type $1,1$-operators is treated systematically in Section 5. In Section 6 the operators resulting from the paradifferential splitting (1.14) is further analysed, especially for their continuity on $\mathscr{S}'(\mathbb{R}^n)$. Estimates in spaces over L_p are discussed in Section 7, including Sobolev and Hölder–Zygmund spaces as special cases of Besov and Lizorkin–Triebel spaces. Section 8 presents a few open problems.

2. PRELIMINARIES ON TYPE $1,1$-OPERATORS

Notation and notions from distribution theory, such as the spaces C_0^∞, $\mathscr{S}$, C^∞ of smooth functions and their duals $\mathscr{D}'$, $\mathscr{S}'$, $\mathscr{E}'$ of distributions, and the Fourier transformation $\mathscr{F}$, will be as in Hörmander's book [Hör85], unless otherwise is mentioned. Eg $\langle u,\,\varphi\rangle$ denotes the value of a distribution u on a test function φ. The space $\mathscr{O}_M(\mathbb{R}^n)$ consists of the slowly increasing $f\in C^\infty(\mathbb{R}^n)$, ie the f that for each multiindex α and some $N>0$ fulfils $|D^\alpha f(x)|\le c(1+|x|)^N$.

As usual $t_+=\max(0,t)$ is the positive part and $[t]$ denotes the greatest integer $\le t$. In general, c will denote a real constant specific to the place of occurrence.

2.1. **The general definition of type $1,1$-operators.** The reader may consult [Joh08b] for an overview of results on type $1,1$-operators and a systematic treatment. The present paper is partly a continuation of [Joh04, Joh05, Joh08b], but it suffices to recall a few facts.

The operators are defined, as usual, on the Schwartz space $\mathscr{S}(\mathbb{R}^n)$ by

$$a(x,D)u=\operatorname{OP}(a)u(x)=(2\pi)^{-n}\int e^{\mathrm{i}x\cdot\eta}a(x,\eta)\mathscr{F}u(\eta)\,d\eta,\qquad u\in\mathscr{S}(\mathbb{R}^n).\tag{2.1}$$

Hereby the symbol $a(x,\eta)$ is required to be in $C^\infty(\mathbb{R}^n\times\mathbb{R}^n)$, of order $d\in\mathbb{R}$ and type $1,1$; ie for all multiindices $\alpha,\,\beta\in\mathbb{N}_0^n$ it fulfils (1.1), or more precisely has finite seminorms

$$p_{\alpha,\beta}(a):=\sup_{x,\eta\in\mathbb{R}^n}(1+|\eta|)^{-(d-|\alpha|+|\beta|)}|D^\alpha_\eta D^\beta_x a(x,\eta)|<\infty.\tag{2.2}$$

The Fréchet space of such symbols is denoted by $S^d_{1,1}(\mathbb{R}^n\times\mathbb{R}^n)$, or just $S^d_{1,1}$. Along with $a(x,D)$ one has the distribution kernel $K(x,y)=\mathscr{F}^{-1}_{\eta\to z}a(x,\eta)\big|_{z=x-y}$, that is C^∞ for $x\neq y$ as usual; cf [Joh08b, Lem. 4.3]. It fulfils $\langle a(x,D)u,\,\varphi\rangle=\langle K,\,\varphi\otimes u\rangle$ for all $u,\,\varphi\in\mathscr{S}$.

For arbitrary $u\in\mathscr{S}'\setminus\mathscr{S}$ it is a delicate question whether or not $a(x,D)u$ is defined. To recall from [Joh08b] how type $1,1$-operators can be defined in general, note that the modified symbol $b(x,\eta)=\psi(2^{-m}D_x)a(x,\eta)$ is given by

$$\hat{b}(\xi,\eta)=\mathscr{F}_{x\to\xi}b(x,\eta)=\psi(2^{-m}\xi)\hat{a}(\xi,\eta).\tag{2.3}$$

Definition 2.1. For a symbol $a(x,\eta)$ in $S^d_{1,1}(\mathbb{R}^n\times\mathbb{R}^n)$ and arbitrary cut-off functions $\psi\in C_0^\infty(\mathbb{R}^n)$ equal to 1 in a neighbourhood of the origin, let

$$a_\psi(x,D)u := \lim_{m\to\infty} \operatorname{OP}(\psi(2^{-m}D_x)a(x,\eta)\psi(2^{-m}\eta))u \tag{2.4}$$

If for each such ψ the limit $a_\psi(x,D)u$ exists in $\mathscr{D}'(\mathbb{R}^n)$ and moreover is independent of ψ, then u belongs to the domain $D(a(x,D))$ by definition and

$$a(x,D)u = a_\psi(x,D)u. \tag{2.5}$$

Thus $a(x,D)$ is a map $\mathscr{S}'(\mathbb{R}^n)\to\mathscr{D}'(\mathbb{R}^n)$ with dense domain.

Since the removal of high frequencies in x and η, that is achieved from $\psi(2^{-m}D_x)$ and $\psi(2^{-m}\eta)$, disappears for $m\to\infty$, this was called definition by *vanishing frequency modulation* in [Joh08b]. (Obviously the action on u is well defined for each m in (2.4) as the modified symbol is in $S^{-\infty}$.) Occasionally the function ψ will be referred to as a *modulation function.*

While the calculus of type 1,1-operators is delicate in general, cf [Hör88, Hör89, Hör97], the following result is straightforward from the definition:

Proposition 2.2. *When $a(x,\eta)$ is in $S^{d_1}_{1,1}(\mathbb{R}^n\times\mathbb{R}^n)$ and $b(\eta)$ belongs to $S^{d_2}_{1,0}(\mathbb{R}^n\times\mathbb{R}^n)$, then $c(x,\eta):=a(x,\eta)b(\eta)$ is in $S^{d_1+d_2}_{1,1}(\mathbb{R}^n\times\mathbb{R}^n)$ and*

$$c(x,D)u = a(x,D)b(D)u, \tag{2.6}$$

where $D(c(x,D)) = D(a(x,D)b(D))$; that is, the two sides are simultaneously defined.

Proof. That $c(x,\eta)$ is in $S^{d_1+d_2}_{1,1}$ can be verified in the usual way from symbolic estimates. For an arbitrary modulation function ψ it is obvious from (2.1) that for every $u\in\mathscr{S}$,

$$\operatorname{OP}(\psi(2^{-m}D_x)a(x,\eta)\psi(2^{-m}\eta))b(D)u = \operatorname{OP}(\psi(2^{-m}D_x)a(x,\eta)\psi(2^{-m}\eta)b(\eta))u. \tag{2.7}$$

This extends to all $u\in\mathscr{S}'$ since the symbols are in $S^{-\infty}$ or $S^{d_2}_{1,0}$. Moreover, for $m\to\infty$ the limit exists on both or none of the two sides for each $u\in\mathscr{S}'$, so in the notation of (2.4),

$$a_\psi(x,D)(b(D)u) = c_\psi(x,D)u. \tag{2.8}$$

Now $u\in D(c(x,D))$ if and only if the right-hand side is independent of ψ, ie if the left-hand side is so, which is equivalent to $b(D)u\in D(a(x,D))$, ie to $u\in D(a(x,D)b(D))$. $\square$

Example 2.3. A standard example of a symbol of type 1,1 results by taking an auxiliary function $A\in C_0^\infty(\mathbb{R}^n)$, say with $\operatorname{supp} A\subset\{\,\eta\mid \frac{3}{4}\le|\eta|\le\frac{5}{4}\,\}$, and $\theta\in\mathbb{R}^n$ fixed:

$$a_\theta(x,\eta) = \sum_{j=0}^{\infty} 2^{jd}e^{-\mathrm{i}2^jx\cdot\theta}A(2^{-j}\eta). \tag{2.9}$$

Clearly $a_\theta\in S^d_{1,1}$ since the terms are disjointly supported.

Such symbols were used by Ching [Chi72] and Bourdaud [Bou88a] for $d=0$, $|\theta|=1$ to show L_2-unboundedness. Refining this, Hörmander [Hör88] linked continuity from H^s with $s>-r$ to the property that θ is a zero of χ of order $r\in\mathbb{N}_0$. Extension to $d\in\mathbb{R}$ was given in [Joh08b].

The non-preservation of wavefront sets discovered by Parenti and Rodino [PR78] was also based on $a_\theta(x,\eta)$. Their ideas were in [Joh08b, Sect. 3.2] extended to all $n\geq 1$ and refined by applying $a_\theta(x,D)$ to a product $v(x)f(x\cdot\theta)$, where $v\in\mathscr{F}^{-1}C_0^\infty$ is an analytic function that controls the spectrum, and the highly oscillating f is Weierstrass' nowhere differentiable function for orders $d\in\,]0,1]$, in a *complex* version with its wavefront set along a half-line. (Nowhere differentiability was shown with a small microlocalisation argument, further explored in [Joh10b].)

Moreover, it was shown in [Joh08b, Lem. 3.2] that $a_\theta(x,D)$ is unclosable in $\mathscr{S}'$ when A is taken to have support in a small neighbourhood of θ. Therefore Definition 2.1 cannot in general be replaced by a closure of the graph in $\mathscr{S}'\times\mathscr{S}'$.

2.1.1. *Action on functions with compact spectra.* As a general result, it was shown in [Joh08b, Sec. 4] that the subspace $\mathscr{S}(\mathbb{R}^n)+\mathscr{F}^{-1}\mathscr{E}'(\mathbb{R}^n)$ always is contained in the domain of $a(x,D)$ and that this is a map

$$a(x,D)\colon \mathscr{S}(\mathbb{R}^n)+\mathscr{F}^{-1}\mathscr{E}'(\mathbb{R}^n)\to\mathscr{O}_M(\mathbb{R}^n). \tag{2.10}$$

In fact, if $u=v+v'$ is an arbitrary splitting of u with $v\in\mathscr{S}$ and $v'\in\mathscr{F}^{-1}\mathscr{E}'$, it was shown that

$$a(x,D)u=a(x,D)v+\operatorname{OP}(a(1\otimes\chi))v', \tag{2.11}$$

whereby $a(1\otimes\chi)(x,\eta)=a(x,\eta)\chi(\eta)$ and $\chi\in C_0^\infty(\mathbb{R}^n)$ is chosen so that $\chi=1$ holds in a neighbourhood of $\operatorname{supp}\mathscr{F}v'$, but otherwise arbitrarily. Here $a(x,\eta)\chi(\eta)$ is in $S^{-\infty}=\bigcap S^d_{1,1}$ so that $\operatorname{OP}(a(1\otimes\chi))$ is defined on $\mathscr{S}'$; and consequently $a(x,D)(\mathscr{F}^{-1}\mathscr{E}')\subset\mathscr{O}_M(\mathbb{R}^n)$.

Remark 2.4. Occasionally it is useful that one can take χ in (2.11) as a cut-off function $\tilde\chi$ fulfilling that $\tilde\chi=1$ only on a neighbourhood of the smaller set

$$\bigcup_{x\in\mathbb{R}^n}\operatorname{supp}a(x,\cdot)\mathscr{F}v'(\cdot). \tag{2.12}$$

Indeed, since $a(1\otimes\chi)\in S^{-\infty}$ it is clear from (2.1) that $\operatorname{OP}(a(1\otimes\chi))v'$ equals $\operatorname{OP}(a(1\otimes\tilde\chi))v'$ at least if $v'\in\mathscr{F}^{-1}C_0^\infty(\mathbb{R}^n)$, but this extends to $v'\in\mathscr{F}^{-1}\mathscr{E}'$ by mollification of $\mathscr{F}v'$.

It is a virtue of (2.10) that $a(x,D)$ is compatible with, say $\operatorname{OP}(S^\infty_{1,0})$. (Compatibility is dicussed in general in [Joh08b].) Therefore some well-known facts extend to type $1,1$-operators:

Example 2.5. Each $a(x,D)$ of type $1,1$ is defined on all polynomials and

$$a(x,D)\Big(\sum_{|\alpha|\leq m}c_\alpha x^\alpha\Big)=\sum_{|\alpha|\leq m}c_\alpha D^\alpha_\eta(e^{\mathrm{i}x\cdot\eta}a(x,\eta))\,\Big|_{\eta=0}\,. \tag{2.13}$$

In fact, since $\hat{f}(\eta)=(2\pi)^n\sum c_\alpha(-D_\eta)^\alpha\delta_0(\eta)$ has support $\{0\}$ it is seen for $v=0$ in (2.10) that $a(x,D)f(x)=\langle\,\hat{f},(2\pi)^{-n}e^{\mathrm{i}\langle x,\cdot\rangle}a(x,\cdot)\chi(\cdot)\,\rangle$ where $\chi=1$ around 0; thence (2.13).

Example 2.6. Also when A is of type $1,1$, one can recover its symbol from the formula

$$a(x,\xi)=e^{-\mathrm{i}x\cdot\xi}A(e^{\mathrm{i}x\cdot\xi}). \tag{2.14}$$

Here $\mathscr{F}e^{\mathrm{i}\langle\cdot,\xi\rangle}=(2\pi)^n\delta_\xi(\eta)$ has compact support, so again it follows from (2.10) that (via a suitable cut-off function) one has $A(e^{\mathrm{i}\langle\cdot,\xi\rangle})=\langle\,\delta_\xi,e^{\mathrm{i}\langle x,\cdot\rangle}a(x,\cdot)\,\rangle=e^{\mathrm{i}x\cdot\xi}a(x,\xi)$.

2.1.2. *Extension to general smooth functions.* To extend $a(x,D)$ to more general sets of smooth functions, it is useful to follow a remark by Bourdaud [Bou88b] on singular integral operators, which shows that every type $1,1$ symbol $a(x,\eta)$ induces a map $\tilde{A}\colon \mathscr{O}_M(\mathbb{R}^n) \to \mathscr{D}'(\mathbb{R}^n)$.

Indeed, Bourdaud defined $\tilde{A}f$ for $f \in \mathscr{O}_M(\mathbb{R}^n)$ as the distribution that on $\varphi \in C_0^\infty(\mathbb{R}^n)$ is given by the following, using the distribution kernel K and an auxiliary function $\chi \in C_0^\infty(\mathbb{R}^n)$ equal to 1 on a neighbourhood of $\operatorname{supp}\varphi$,

$$\langle \tilde{A}f, \varphi\rangle = \langle a(x,D)(\chi f), \varphi\rangle + \iint K(x,y)(1-\chi(y))f(y)\varphi(x)\,dy\,dx. \tag{2.15}$$

However, one may restate this in terms of the tensor product $1\otimes f$ in $\mathscr{S}'(\mathbb{R}^n\times\mathbb{R}^n)$ acting on $(\varphi\otimes(1-\chi))K \in \mathscr{S}(\mathbb{R}^n\times\mathbb{R}^n)$, ie

$$\langle \tilde{A}f, \varphi\rangle = \langle a(x,D)(\chi f), \varphi\rangle + \langle 1\otimes f, (\varphi\otimes(1-\chi))K\rangle, \tag{2.16}$$

The advantage here is that both terms obviously makes sense as long as f is smooth and temperate, ie for every $f \in C^\infty(\mathbb{R}^n)\bigcap\mathscr{S}'(\mathbb{R}^n)$.

More precisely, for φ with support in the interior $\mathscr{C}^\circ$ of a compact set $\mathscr{C}\subset\mathbb{R}^n$ and $\chi = 1$ on a neighbourhood of $\mathscr{C}$, the right-hand side of (2.16) gives the same value for any $\tilde{\chi}\in C_0^\infty$ equal to 1 around $\mathscr{C}$, for after subtraction the kernel relation implies that $\langle a(x,D)((\chi-\tilde{\chi})f), \varphi\rangle$ has sign opposite to that of $\langle 1\otimes f, (\varphi(\tilde{\chi}-\chi))K\rangle$. Crude estimates now show that $\tilde{A}f$ yields a distribution in $\mathscr{D}'(\mathscr{C}^\circ)$, and the χ-independence implies that it coincides in $\mathscr{D}'(\mathscr{C}^\circ\bigcap\mathscr{C}_1^\circ)$ with the distribution defined from another compact set $\mathscr{C}_1$. Since $\mathbb{R}^n = \bigcup\mathscr{C}^\circ$, the *recollement de morceaux* theorem yields that a distribution $\tilde{A}f \in \mathscr{D}'(\mathbb{R}^n)$ is defined by (2.16).

In relation to Definition 2.1, the above gives the point of departure for the new result that $a(x,D)$ always is a map defined on the *maximal* set of smooth functions, ie on $C^\infty\bigcap\mathscr{S}'$:

Theorem 2.7. *Every $a(x,D)\in \mathrm{OP}(S^d_{1,1}(\mathbb{R}^n\times\mathbb{R}^n))$ restricts to a map*

$$a(x,D)\colon C^\infty(\mathbb{R}^n)\bigcap\mathscr{S}'(\mathbb{R}^n)\to C^\infty(\mathbb{R}^n), \tag{2.17}$$

which is given by (2.16) *and maps the subspace $\mathscr{O}_M(\mathbb{R}^n)$ into itself.*

Proof. Let $A_m = \mathrm{OP}(\psi(2^{-m}D_x)a(x,\eta)\psi(2^{-m}\eta))$ with kernel K_m, so $a(x,D)u = \lim_m A_m u$ when $u\in D(a(x,D))$. With $f\in C^\infty\bigcap\mathscr{S}'$ and φ,χ as above, this is the case for $u=\chi f\in C_0^\infty$, and since the support of $\varphi\otimes(1-\chi)$ is disjoint from the diagonal and bounded in the x-direction, [Joh08b, Prop. 6.1] asserts that in the topology of $\mathscr{S}(\mathbb{R}^n\times\mathbb{R}^n)$

$$\varphi(x)(1-\chi(y))K_m(x,y) \xrightarrow[m\to\infty]{} \varphi(x)(1-\chi(y))K(x,y). \tag{2.18}$$

Exploiting these facts in (2.16) yields that

$$\langle \tilde{A}f, \varphi\rangle = \lim_m \langle A_m(\chi f), \varphi\rangle + \lim_m \iint K_m(x,y)(1-\chi(y))f(y)\varphi(x)\,dy\,dx. \tag{2.19}$$

Here the integral equals $\langle A_m(f-\chi f), \varphi\rangle$ by the kernel relation, for $A_m\in\mathrm{OP}(S^{-\infty})$ and f may as an element of $\mathscr{S}'$ be approached from C_0^∞. So (2.19) yields

$$\langle \tilde{A}f, \varphi\rangle = \lim_m \langle A_m(\chi f), \varphi\rangle + \lim_m\langle A_m(f-\chi f), \varphi\rangle = \lim_m\langle A_m f, \varphi\rangle. \tag{2.20}$$

Thus $A_m f\to\tilde{A}f$, which is independent of ψ. Hence $\tilde{A}\subset a(x,D)$ as desired.

Moreover, $\tilde{A}f$ is smooth because $a(x,D)(f\chi) \in \mathscr{S}$ while the other contribution in (2.16) also acts like a C^∞-function: when $\tilde{\varphi} \in C_0^\infty$ is chosen to be 1 around $\operatorname{supp}\varphi$ and supported by $\mathscr{C}^\circ$, cf the construction of $\tilde{A}f$, then the second term equals

$$\int \langle\, f, (\tilde{\varphi}(x)(1-\chi))K(x,\cdot)\,\rangle \varphi(x)\,dx, \tag{2.21}$$

where $x \mapsto \langle\, f, \tilde{\varphi}(x)(1-\chi(\cdot))K(x,\cdot)\,\rangle$ is C^∞ as seen in the verification that $g \otimes f \in \mathscr{S}'(\mathbb{R}^n \times \mathbb{R}^n)$ for $f, g \in \mathscr{S}'(\mathbb{R}^n)$. Therefore $\tilde{A}f$ is locally smooth, so $\tilde{A}f \in C^\infty(\mathbb{R}^n)$ follows.

When in addition $f \in \mathscr{O}_M$, then $(1+|x|)^{-2N}D^\alpha\tilde{A}f$ is bounded for sufficiently large N, for when $r = \operatorname{dist}(\operatorname{supp}\varphi, \operatorname{supp}(1-\chi))$ one finds in the second contribution to (2.15) that

$$\begin{aligned}(1+|y|)^{2N}|D_x^\alpha K(x,y)| &\le (1+|x|)^{2N}\max(1,1/r)^{2N}(r+|x-y|)^{2N}|D_x^\alpha K(x,y)| \\ &\le c(1+|x|)^{2N}\sup_{x\in\mathbb{R}^n}\int |D_x^\alpha(2\Delta_\eta)^N a(x,\eta)|\,d\eta,\end{aligned} \tag{2.22}$$

where the supremum is finite for $2N > d+|\alpha|+n$ whilst $(1+|y|)^{-2N}f(y)$ is in L_1 for large N. Hence $\tilde{A}f \in \mathscr{O}_M$ as claimed. $\square$

In view of the theorem, the difficulties for type $1,1$-operators are unrelated to growth at infinity for C^∞-functions. Moreover, the codomain C^∞ in Theorem 2.7 is not contained in $\mathscr{S}'$, but this is consistent with $\mathscr{D}'$ as the codomain in Definition 2.1.

Example 2.8. The space $C^\infty(\mathbb{R}^n)\bigcap\mathscr{S}'(\mathbb{R}^n)$ clearly contains functions of non-slow growth, eg

$$f(x) = e^{x_1+\dots+x_n}\cos(e^{x_1+\dots+x_n}). \tag{2.23}$$

Recall that $f \in \mathscr{S}'$ because $f = \mathrm{i}D_1 g$ for $g(x) = \sin(e^{x_1+\dots+x_n})$, which is in $L_\infty \subset \mathscr{S}'$. But $g \notin \mathscr{O}_M$, so already for $a(x,D) = \mathrm{i}D_1$ the space $\mathscr{O}_M$ cannot contain the range in Theorem 2.7.

Remark 2.9. In remarks prior to the proof of the $T1$-theorem, it was explained in [DJ84] that just a few properties of the distribution kernel of a continuous map $T\colon C_0^\infty(\mathbb{R}^n) \to \mathscr{D}'(\mathbb{R}^n)$ implies that $T(1)$ is well defined modulo constants. In particular this was applied to $T \in \operatorname{OP}(S^0_{1,1})$, but in that case their extension is equal to the above of Bourdaud, so according to Theorem 2.7 it also gives the same result as Definition 2.1.

2.2. **Conditions along the twisted diagonal.** As the first explicit condition on the symbol of a type $1,1$-operator, Hörmander [Hör88] proved that $a(x,D)$ has an extension by continuity

$$H^{s+d} \to H^s \quad \text{for every } s \in \mathbb{R} \tag{2.24}$$

whenever $a \in S^d_{1,1}(\mathbb{R}^n \times \mathbb{R}^n)$ fulfils the *twisted diagonal condition*: for some $B \ge 1$

$$\hat{a}(\xi,\eta) = 0 \quad \text{where} \quad B(1+|\xi+\eta|) < |\eta|. \tag{2.25}$$

This means that the partially Fourier transformed symbol $\hat{a}(\xi,\eta) := \mathscr{F}_{x\to\xi}a(x,\eta)$ vanishes in a conical neighbourhood of a non-compact part of the twisted diagonal

$$\mathscr{T} = \{\,(\xi,\eta) \in \mathbb{R}^n \times \mathbb{R}^n \mid \xi+\eta = 0\,\}. \tag{2.26}$$

Localisations to conical neighbourhoods of $\mathscr{T}$ was also introduced by Hörmander in [Hör88, Hör89, Hör97]. Specifically this meant to pass from $a(x,\eta)$ to $a_{\chi,\varepsilon}(x,\eta)$ defined by

$$\hat{a}_{\chi,\varepsilon}(\xi,\eta)=\hat{a}(\xi,\eta)\chi(\xi+\eta,\varepsilon\eta), \tag{2.27}$$

whereby $\chi\in C^\infty(\mathbb{R}^n\times\mathbb{R}^n)$ is chosen so that

$$\chi(t\xi,t\eta)=\chi(\xi,\eta)\quad\text{for}\quad t\geq 1,\ |\eta|\geq 2 \tag{2.28}$$

$$\operatorname{supp}\chi\subset\{\,(\xi,\eta)\mid 1\leq|\eta|,\ |\xi|\leq|\eta|\,\} \tag{2.29}$$

$$\chi=1\quad\text{in}\quad\{\,(\xi,\eta)\mid 2\leq|\eta|,\ 2|\xi|\leq|\eta|\,\}. \tag{2.30}$$

Using this, Hörmander analysed a milder condition than the strict vanishing in (2.25), namely that for some $\sigma\in\mathbb{R}$, it holds for all multiindices α and $0<\varepsilon<1$ that

$$N_{\chi,\varepsilon,\alpha}(a):=\sup_{R>0,\,x\in\mathbb{R}^n}R^{-d}\Big(\int_{R\leq|\eta|\leq 2R}|R^{|\alpha|}D^\alpha_\eta a_{\chi,\varepsilon}(x,\eta)|^2\,\frac{d\eta}{R^n}\Big)^{1/2}\leq c_{\alpha,\sigma}\varepsilon^{\sigma+n/2-|\alpha|}. \tag{2.31}$$

This asymptotics for $\varepsilon\to 0$ always holds for $\sigma=0$, as was proved in [Hör97, Lem. 9.3.2]:

Lemma 2.10. *When $a\in S^d_{1,1}(\mathbb{R}^n\times\mathbb{R}^n)$ and $0<\varepsilon\leq 1$, then $a_{\chi,\varepsilon}\in C^\infty$ and*

$$|D^\alpha_\eta D^\beta_x a_{\chi,\varepsilon}(x,\eta)|\leq C_{\alpha,\beta}(a)\varepsilon^{-|\alpha|}(1+|\eta|)^{d-|\alpha|+|\beta|} \tag{2.32}$$

$$\Big(\int_{R\leq|\eta|\leq 2R}|D^\alpha_\eta a_{\chi,\varepsilon}(x,\eta)|^2\,d\eta\Big)^{1/2}\leq C_\alpha R^d(\varepsilon R)^{n/2-|\alpha|}. \tag{2.33}$$

The map $a\mapsto a_{\chi,\varepsilon}$ is continuous in $S^d_{1,1}$.

The last remark on continuity has been inserted here for later reference. It is easily verified by observing in the proof of [Hör97, Lem. 9.3.2] (to which the reader is referred) that the constant $C_{\alpha,\beta}(a)$ is a continuous seminorm in $S^d_{1,1}$.

For $\sigma>0$ the faster convergence to 0 in (2.31) was proved to imply boundedness

$$a(x,D)\colon H^{s+d}(\mathbb{R}^n)\to H^s(\mathbb{R}^n)\quad\text{for}\quad s>-\sigma. \tag{2.34}$$

The reader could consult [Hör97, Thm. 9.3.5] for this (and [Hör97, Thm. 9.3.7] for four pages of proof of necessity of $s\geq-\sup\sigma$, with supremum over all σ for which (2.31) holds).

If $\hat{a}(\xi,\eta)$ is so small along $\mathscr{T}$ that (2.31) holds for all $\sigma\in\mathbb{R}$, consequently there is boundedness $H^{s+d}\to H^s$ for all $s\in\mathbb{R}$. Eg this is the case when (2.25) holds, for since

$$\operatorname{supp}\hat{a}_{\chi,\varepsilon}\subset\{\,(\xi,\eta)\mid 1+|\xi+\eta|\leq 2\varepsilon|\eta|\,\}, \tag{2.35}$$

clearly $a_{\chi,\varepsilon}\equiv 0$ for $2\varepsilon>1/B$ then.

Example 2.11. For the present paper it is interesting to use Ching's symbol (2.9) to show the existence of symbols fulfilling (2.31) for a given $\sigma\in\mathbb{N}$. To do so one may fix $|\theta|=1$ and take some $A(\eta)$ in $C^\infty_0(\{\,\eta\mid\frac{3}{4}<|\eta|<\frac{5}{4}\,\})$ with a zero of order σ at θ, so that Taylor's formula gives $|A(\eta)|\leq c|\eta-\theta|^\sigma$ in a neighbourhood of θ.

As $\hat{a}(x,\eta) = (2\pi)^n \sum_{j=0}^\infty 2^{jd}\delta(\xi+2^j\theta)A(2^{-j}\eta)$, clearly

$$a_{\theta,\chi,\varepsilon}(x,\eta) = \sum_{j=0}^\infty 2^{jd}e^{-\mathrm{i}x\cdot 2^j\theta}\chi(\eta-2^j\theta,\varepsilon\eta)A(2^{-j}\eta). \tag{2.36}$$

Because $[R,2R]$ is contained in $[\frac{3}{4}2^{j-1},\frac{3}{2}2^{j-1}]\cup[\frac{3}{4}2^j,\frac{3}{2}2^j]$ for some $j\in\mathbb{Z}$, it suffices to estimate the integral in (2.31) only for $R=3\cdot 2^{j-2}$ with $j\geq 1$. Then it involves only the jth term, ie

$$\int_{R\leq|\eta|\leq 2R}|a_{\theta,\chi,\varepsilon}(x,\eta)|^2\,d\eta = \int_{R\leq|\eta|\leq 2R}R^{2d}|A(\eta/R)|^2|\chi(\eta-R\theta,\varepsilon\eta)|^2\,d\eta. \tag{2.37}$$

By the choice of χ, the integrand is 0 unless $|\eta-R\theta|\leq\varepsilon|\eta|\leq 2\varepsilon R$ and $1\leq\varepsilon R$, so for small ε,

$$\int_{R\leq|\eta|\leq 2R}|a_{\theta,\chi,\varepsilon}(x,\eta)|^2\,d\eta \leq \|\chi\|_\infty^2 R^{n+2d}\int_{|\zeta-\theta|\leq 2\varepsilon}(c|\zeta-\theta|^\sigma)^2\,d\zeta \leq c'\varepsilon^{2\sigma+n}R^{n+2d}. \tag{2.38}$$

Applying $(RD_\eta)^\alpha$ before integration, $(RD_\eta)^\gamma$ may fall on $A(\eta/R)$, which lowers the degree and yields (at most) $\varepsilon^{n/2+\sigma-|\gamma|}$. In the factor $(RD_\eta)^{\alpha-\gamma}\chi(\eta-R\theta,\varepsilon\eta)$ the homogeneity of degree $-|\alpha-\gamma|$ applies for $\varepsilon R\geq 2$ and yields a bound in terms of finite suprema over $B(\theta,2)\times B(0,2)$, hence is $\mathscr{O}(1)$; else $\varepsilon R<2$ so the factor is $\mathscr{O}(R^{|\alpha-\gamma|})=\mathscr{O}(\varepsilon^{|\gamma|-|\alpha|})$ when non-zero, as both entries are in norm less than 4 then. Altogether this verifies (2.31). — A lower bound of (2.37) by $c\varepsilon^{2\sigma+n}R^{n+2d}$ is similar (cf [Hör97, Ex. 9.3.3] for $\sigma=0=d$) when $|A(\eta)|\geq c_0|\eta-\theta|^\sigma$, which can be obtained by taking A as a localisation of the right-hand side for *even* σ (so $A\in C^\infty$); and this shows that (2.31) cannot hold for larger values of σ for this choice of $a_\theta(x,\eta)$.

3. Pointwise estimates

A crucial technique in this paper will be to estimate $|a(x,D)u(x)|$ at an arbitrary point of $\mathbb{R}^n$. The recent results on this by the author [Joh10a] are recalled here and further elaborated in Section 3.2 with an estimate of frequency modulated operators.

3.1. **The factorisation inequality.**

First of all, by [Joh10a, Thm. 4.1], when $\operatorname{supp}\hat{u}$ is compact in $\mathbb{R}^n$, the action on u by $a(x,D)$ can be *separated* from u at the cost of an estimate, which is the *factorisation inequality*

$$|a(x,D)u(x)|\leq F_a(N,R;x)u^*(N,R;x). \tag{3.1}$$

Here u^* denotes the maximal function of Peetre–Fefferman–Stein type, defined as

$$u^*(N,R;x)=\sup_{y\in\mathbb{R}^n}\frac{|u(x-y)|}{(1+R|y|)^N}=\sup_{y\in\mathbb{R}^n}\frac{|u(y)|}{(1+R|x-y|)^N} \tag{3.2}$$

when $\operatorname{supp}\hat{u}\subset\overline{B}(0,R)$. The parameter N may eg be chosen so that $N\geq\operatorname{order}\hat{u}$.

The a-factor F_a, also called the symbol factor, only depends on u in a vague way, viz. through N and R. It is related to the distribution kernel of $a(x,D)$. More precisely

$$F_a(N,R;x)=\int_{\mathbb{R}^n}(1+R|y|)^N|\mathscr{F}^{-1}_{\eta\to y}(a(x,\eta)\chi(\eta))|\,dy, \tag{3.3}$$

where $\chi\in C_0^\infty(\mathbb{R}^n)$ should equal 1 on a neighbourhood of $\operatorname{supp}\hat{u}$ (or of $\bigcup_x\operatorname{supp}a(x,\cdot)\hat{u}(\cdot)$).

The estimate (3.1) is useful as both factors are easily controlled. Eg $u^*(x)$ is polynomially bounded, for $|u(y)| \le c(1+|y|)^N \le c(1+R|y-x|)^N(1+|x|)^N$ holds according to the Paley–Wiener–Schwartz Theorem if $N \ge \operatorname{order} \hat{u}$, $R \ge 1$, and by (3.2) this implies

$$u^*(N,R;x) \le c(1+|x|)^N, \qquad x \in \mathbb{R}^n. \tag{3.4}$$

Here it is first recalled that every $u \in \mathscr{S}'$ has finite order as its value $\langle u,\, \psi \rangle$ on $\psi \in \mathscr{S}$ fulfils

$$|\langle u,\, \psi \rangle| \le c p_N(\psi), \tag{3.5}$$

$$p_N(\psi) = \sup\{\, (1+|x|)^N |D^\alpha u(x)| \mid x \in \mathbb{R}^n,\ |\alpha| \le N \,\}. \tag{3.6}$$

Indeed, for $\psi = \varphi \in C_0^\infty$ an estimate of $(1+|x|)^N$ on $\operatorname{supp}\varphi$ shows that u is of order N. To avoid the discussion whether the converse is true, it will throughout be convenient to call the least integer N fulfilling (3.5) the *temperate* order of u, written $N = \operatorname{order}_{\mathscr{S}'}(u)$.

Returning to (3.4), when the compact spectrum of u results from Fourier multiplication, then the below $\mathscr{O}(2^{kN})$-information on the constant will be used repeatedly in the present paper.

Lemma 3.1. *Let $u \in \mathscr{S}'(\mathbb{R}^n)$ be arbitrary and $N \ge \operatorname{order}_{\mathscr{S}'}(\hat{u})$. When $\psi \in C_0^\infty(\mathbb{R}^n)$ has support in $\overline{B}(0,R)$, then $w = \psi(2^{-k}D)u$ fulfils*

$$w^*(N,R2^k;x) \le C2^{kN}(1+|x|)^N, \qquad k \in \mathbb{N}_0, \tag{3.7}$$

for a constant C independent of k.

Proof. Since $\psi(2^{-k}D)u(x) = \langle \hat{u},\, \psi(2^{-k}\cdot)e^{\mathrm{i}\langle x,\cdot\rangle}(2\pi)^{-n} \rangle$, continuity of $\hat{u}\colon \mathscr{S} \to \mathbb{C}$ yields

$$|w(x)| \le c \sup\big\{\, (1+|\xi|)^N |D_\xi^\alpha(\psi(2^{-k}\xi)e^{\mathrm{i}\langle x,\xi\rangle})| \,\big|\, |\alpha| \le N, \quad \xi \in \mathbb{R}^n \,\big\}. \tag{3.8}$$

As $|(1+|\xi|)^N D^\alpha \psi(2^{-k}\xi)| \le c' 2^{k(N-|\alpha|)}$, Leibniz' rule yields that $|w(x)| \le c'' 2^{kN}(1+|x|)^N$. Proceeding as before the lemma, the inequality follows with $C = c'' \max(1, R^{-N})$. $\square$

The non-linear map $u \mapsto u^*$ is also bounded with respect to the L_p-norm, which can be shown in an elementary way; cf [Joh10a, Thm. 2.6].

Secondly, for the a-factor one has $F_a \in C(\mathbb{R}^n) \cap L_\infty(\mathbb{R}^n)$ with estimates highly reminiscent of the Mihlin–Hörmander conditions for Fourier multipliers:

Theorem 3.2. *Assume the symbol $a(x,\eta)$ is in $S^d_{1,1}(\mathbb{R}^n \times \mathbb{R}^n)$ and let $F_a(N,R;x)$ be given by* (3.3) *for parameters $R, N > 0$, with the auxiliary function taken as $\chi = \psi(R^{-1}\cdot)$ for $\psi \in C_0^\infty(\mathbb{R}^n)$ equalling* 1 *in a set with non-empty interior. Then it holds for all $x \in \mathbb{R}^n$ that*

$$0 \le F_a(x) \le c_{n,N} \sum_{|\alpha| \le N + [\frac{n}{2}] + 1} \Big(\int_{R \operatorname{supp}\psi} |R^{|\alpha|} D_\eta^\alpha a(x,\eta)|^2 \, \frac{d\eta}{R^n} \Big)^{1/2}. \tag{3.9}$$

For the elementary proof of this the reader is referred to [Joh10a]; cf Theorem 4.1 and Section 6 there. A further analysis of the dependence on $a(x,\eta)$ and R was given in [Joh10a, Cor. 4.6]:

Corollary 3.3. *Assume $a \in S^d_{1,1}(\mathbb{R}^n \times \mathbb{R}^n)$ and let N, R and ψ be as in Theorem 3.2. When $R \geq 1$ there is a seminorm p on $S^d_{1,1}$ and a constant $c > 0$, depending only on n, N and ψ, such that*

$$0 \leq F_a(x) \leq c p(a) R^{\max(d, N+[n/2]+1)} \quad \textit{for all} \quad x \in \mathbb{R}^n. \tag{3.10}$$

Moreover, if $\operatorname{supp}\psi$ *is contained in a corona $\{\, \eta \mid \theta_0 \leq |\eta| \leq \Theta_0 \,\}$, and $\psi(\eta) = 1$ holds for $\theta_1 \leq |\eta| \leq \Theta_1$, whereby $0 \neq \theta_0 < \theta_1 < \Theta_1 < \Theta_0$, then*

$$0 \leq F_a(x) \leq c_0 p(a) R^d \quad \textit{for all} \quad x \in \mathbb{R}^n, \tag{3.11}$$

with $c_0 = c \max(1, \theta_0^{d-N-[n/2]-1}, \theta_0^d)$.

The above asymptotics for $R \to \infty$ is $\mathscr{O}(R^d)$ for large d. This can be improved when $a(x,\eta)$ has been subjected to modulation of the frequencies in the x-variable. With a second spectral quantity $Q > 0$, the following was shown in [Joh10a, Cor. 4.8], cf Section 6 there:

Corollary 3.4. *When $a_Q(x,\eta) = \varphi(Q^{-1}D_x)a(x,\eta)$ for some $a \in S^d_{1,1}(\mathbb{R}^n \times \mathbb{R}^n)$ and $\varphi \in C_0^\infty(\mathbb{R}^n)$ with $\varphi = 0$ in a neighbourhood of $\xi = 0$, then there is a seminorm p on $S^d_{1,1}$ and constants c_M, depending only on M, n, N, ψ and φ, such that*

$$0 \leq F_{a_Q}(N,R;x) \leq c_M p(a) Q^{-M} R^{\max(d+M, [N+n/2]+1)} \quad \textit{for} \quad M, Q, R > 0. \tag{3.12}$$

Here $d + M$ can replace the maximum when the auxiliary function ψ in F_{a_Q} fulfils the corona condition in Corollary 3.3.

Remark 3.5. The proof in [Joh10a] shows that the seminorm in Corollary 3.3 may be chosen in the same way for all d, namely $p(a) = \sum_{|\alpha| \leq [N+n/2]+1} p_{\alpha,0}(a)$; cf (2.2). Similarly for Corollary 3.4.

3.2. **Estimates of frequency modulated operators.** The results in the previous section easily give the following, which is used repeatedly later in Sections 5 and 6.

Proposition 3.6. *For $a(x,\eta)$ in $S^d_{1,1}(\mathbb{R}^n \times \mathbb{R}^n)$, $u, v \in \mathscr{S}'(\mathbb{R}^n)$ and arbitrary Φ, $\Psi \in C_0^\infty(\mathbb{R}^n)$, for which Ψ is constant in a neighbourhood of the origin and is supported by $\overline{B}(0,R)$ for $R \geq 1$, there is a $c > 0$ which for $k \in \mathbb{N}_0$ and $N \geq \operatorname{order}_{\mathscr{S}'}(\mathscr{F}v)$ gives the polynomial bound,*

$$\left| \operatorname{OP}\left(\Phi(2^{-k}D_x)a(x,\eta)\Psi(2^{-k}\eta)\right)v(x) \right| \leq c 2^{k(N+d)_+}(1+|x|)^N. \tag{3.13}$$

Here the positive part $(\cdot)_+ = \max(0,\cdot)$ is redundant when $0 \notin \operatorname{supp}\Psi$.

Proof. For the purposes of this proof it is convenient to let $a^k(x,\eta) = \Phi(2^{-k}D_x)a(x,\eta)$ and $v^k = \Psi(2^{-k}D)v$. By the factorisation inequality (3.1), there is an estimate in terms of a product

$$|a^k(x,D)v^k(x)| \leq F_{a^k}(N, R2^k; x) \cdot (v^k)^*(N, R2^k; x). \tag{3.14}$$

Here, for $N \geq \operatorname{order}_{\mathscr{S}'}(\hat{v})$, Lemma 3.1 asserts that $(v^k)^*(N, R2^k; x) \leq C 2^{kN}(1+|x|)^N$ for $x \in \mathbb{R}^n$.

When $0 \notin \operatorname{supp}\Psi$, the auxiliary function $\chi = \psi(\cdot/(R2^k))$ used in F_{a^k}, cf Theorem 3.2, can be so chosen that it fulfils the corona condition in Corollary 3.3. Since Remark 3.5 implies $p(a^k) \leq p(a) \int |\mathscr{F}^{-1}\Phi(y)|\,dy$, there is by Corollary 3.4 with $Q = 2^{-k}$ an estimate

$$0 \leq F_{a^k}(N, R2^k; x) \leq c_1 \|\mathscr{F}^{-1}\Phi\|_1 p(a) 2^{kd}, \tag{3.15}$$

where c_1 depends on n, N only. These inequalities yield the claim in this case.

In general $v^k = v_k + v_{k-1} + \dots + v_1 + v^0$, whereby v_j denotes the difference $v^j - v^{j-1} = \Psi(2^{-j}D)v - \Psi(2^{-j+1}D)v$. This gives the starting point

$$|a^k(x,D)v^k(x)| \le |a^k(x,D)v^0(x)| + \sum_{j=1}^{k} F_{a^k}(N,R2^j;x)v_j^*(N,R2^j;x). \tag{3.16}$$

As $\tilde{\Psi} = \Psi - \Psi(2\cdot)$ does not have 0 in its support, the above shows that with the same c_1 one has $F_{a^k}(N,R2^j;x) \le c_1\|\mathscr{F}^{-1}\Phi\|_1 p(a)2^{jd}$ for $j = 1,\dots,k$. Lemma 3.1 also yields control of v_j^*, so the sum on the right-hand side of (3.16) is estimated, for $d+N \ne 0$, by

$$\sum_{j=1}^{k} c_1C'p(a)2^{j(N+d)}(1+|x|)^N \le c_1C'p(a)(1+|x|)^N 2^{(k+1)(N+d)_+}. \tag{3.17}$$

This upper bound extends to $d = -N$ because $a \in S^{d+\varepsilon}_{1,1}$ for $\varepsilon > 0$, for c_1 is constant with respect to d whilst Remark 3.5 gives that the seminorm in $S^{d+\varepsilon}_{1,1}$ is $\le p(a)$ as $\varepsilon \to 0$.

The remainder in (3.16) fulfils $|a^k(x,D)v^0(x)| \le c_1R^{N'}(1+|x|)^N$ for a large N'; cf the first part of Corollary 3.3 and Lemma 3.1. Altogether $|a^k(x,D)v^k(x)| \le c2^{k(N+d)_+}(1+|x|)^N$. $\square$

4. ADJOINTS OF TYPE $1,1$-OPERATORS

4.1. **The basic lemma.** To show that the twisted diagonal condition (2.25) also implies continuity $a(x,D)\colon \mathscr{S}' \to \mathscr{S}'$, a lemma on the adjoint symbols is recalled. It was proved in [Hör88] and [Hör97, Lem. 9.4.1], but given here in a slightly more precise form.

Lemma 4.1. *When $a(x,\eta)$ is in $S^d_{1,1}(\mathbb{R}^n\times\mathbb{R}^n)$ and for some $B\ge 1$ satisfies the twisted diagonal condition* (2.25)*, then the adjoint $a(x,D)^* = b(x,D)$ has the symbol $b(x,\eta) = e^{\mathrm{i}D_x\cdot D_\eta}\overline{a(x,\eta)}$, which is in $S^d_{1,1}(\mathbb{R}^n\times\mathbb{R}^n)$ in this case and*

$$\hat{b}(\xi,\eta) = 0 \quad \textit{when} \quad |\xi+\eta| > B(|\eta|+1). \tag{4.1}$$

Moreover,

$$|D^\alpha_\eta D^\beta_x b(x,\eta)| \le C_{\alpha\beta}(a)B(1+B^{d-|\alpha|+|\beta|})(1+|\eta|)^{d-|\alpha|+|\beta|}, \tag{4.2}$$

for certain continuous seminorms $C_{\alpha\beta}$ on $S^d_{1,1}(\mathbb{R}^n\times\mathbb{R}^n)$, that do not depend on B.

In view of the lemma, if $a(x,D)$ fulfils the twisted diagonal condition (2.25), it obviously has the continuous linear *extension* $b(x,D)^*\colon \mathscr{S}'(\mathbb{R}^n) \to \mathscr{S}'(\mathbb{R}^n)$. But it still has to be shown that this coincides with the definition of $a(x,D)$ by vanishing frequency modulation:

Proposition 4.2. *When $a(x,\eta) \in S^d_{1,1}(\mathbb{R}^n\times\mathbb{R}^n)$ fulfils* (2.25)*, then $a(x,D)$ is a continuous linear map $\mathscr{S}'(\mathbb{R}^n) \to \mathscr{S}'(\mathbb{R}^n)$ and it equals the adjoint of $b(x,D)\colon \mathscr{S}(\mathbb{R}^n) \to \mathscr{S}(\mathbb{R}^n)$, when $b(x,\eta)$ is the adjoint symbol as in Lemma 4.1.*

Proof. When $\psi \in C_0^\infty(\mathbb{R}^n)$ is such that $\psi = 1$ in a neighbourhood of the origin, a simple convolution estimate (cf [Joh08b, Lem. 2.1]) gives that in the topology of $S^{d+1}_{1,1}$,

$$\psi(2^{-m}D_x)a(x,\eta)\psi(2^{-m}\eta) \to a(x,\eta) \quad \text{for} \quad m \to \infty. \tag{4.3}$$

Since the supports of the partially Fourier transformed symbols

$$\psi(2^{-m}\xi)\mathscr{F}_{x\to\xi}a(\xi,\eta)\psi(2^{-m}\eta), \quad m \in \mathbb{N}, \tag{4.4}$$

are contained in $\operatorname{supp}\mathscr{F}_{x\to\xi}a(\xi,\eta)$, clearly this sequence also fulfils (2.25) for the same B. As the passage to adjoint symbols by (4.2) is continuous from the metric subspace of $S^d_{1,1}$ fulfilling (2.25) to $S^{d+1}_{1,1}$, one therefore has that

$$b_m(x,\eta) := e^{\mathrm{i}D_x\cdot D_\eta}(\overline{\psi(2^{-m}D_x)a(x,\eta)\psi(2^{-m}\eta)}) \xrightarrow[m\to\infty]{} e^{\mathrm{i}D_x\cdot D_\eta}\overline{a(x,\eta)} =: b(x,\eta). \tag{4.5}$$

Combining this with the fact that $b(x,D)$ as an operator on the Schwartz space depends continuously on the symbol, one has for $u \in \mathscr{S}'(\mathbb{R}^n)$, $\varphi \in \mathscr{S}(\mathbb{R}^n)$,

$$\begin{aligned} (\,b(x,D)^*u\,|\,\varphi\,) &= (\,u\,|\lim_{m\to\infty}\mathrm{OP}(b_m(x,\eta))\varphi\,) \\ &= \lim_{m\to\infty}(\,\mathrm{OP}(\psi(2^{-m}D_x)a(x,\eta)\psi(2^{-m}\eta))u\,|\,\varphi\,). \end{aligned} \tag{4.6}$$

As the left-hand side is independent of ψ the limit in (2.4) is so, hence the definition of $a(x,D)$ gives that every $u \in \mathscr{S}'(\mathbb{R}^n)$ is in $D(a(x,D))$ and $a(x,D)u = b(x,D)^*u$ as claimed. □

The mere extendability to $\mathscr{S}'$ under the twisted diagonal condition (2.25) could have been observed already in [Hör88, Hör97], but the above result seems to be the first giving a sufficient condition for a type $1,1$-operator to be *defined* on the entire $\mathscr{S}'(\mathbb{R}^n)$.

4.2. **The self-adjoint subclass $\tilde{\mathbf{S}}^{\mathbf{d}}_{\mathbf{11}}$.** Proposition 4.2 shows that the twisted diagonal condition (2.25) suffices for $D(a(x,D)) = \mathscr{S}'(\mathbb{R}^n)$, but this condition is too strong to be necessary. A vanishing to infinite order along $\mathscr{T}$ should suffice.

In this section, the purpose is to prove that $a(x,D)\colon \mathscr{S}' \to \mathscr{S}'$ is continuous if more generally the twisted diagonal condition of order σ, that is (2.31), holds for all $\sigma \in \mathbb{R}$.

This will supplement Hörmander's investigation in [Hör88, Hör89, Hör97], from where the main ingredients are recalled. Using (2.27) and $\mathscr{F}_{x\to\xi}$ it follows that in $\mathscr{S}'(\mathbb{R}^n \times \mathbb{R}^n)$,

$$a(x,\eta) = (a(x,\eta) - a_{\chi,1}(x,\eta)) + \sum_{\nu=0}^{\infty}(a_{\chi,2^{-\nu}}(x,\eta) - a_{\chi,2^{-\nu-1}}(x,\eta)). \tag{4.7}$$

Here the term $a(x,\eta) - a_{\chi,1}(x,\eta)$ fulfils (2.25) for $B = 1$, so Proposition 4.2 applies to it.

Introducing $e_\varepsilon(x,D)$ as in [Hör97, Sect. 9.3] as

$$\hat{e}_\varepsilon(x,\eta) = \hat{a}_{\chi,\varepsilon}(\xi,\eta) - \hat{a}_{\chi,\varepsilon/2}(\xi,\eta) = (\chi(\xi+\eta,\varepsilon\eta) - \chi(\xi+\eta,\varepsilon\eta/2))\hat{a}(x,\eta), \tag{4.8}$$

it is useful to infer from the choice of χ that

$$\operatorname{supp}\hat{e}_\varepsilon \subset \{\,(\xi,\eta)\;\big|\;\tfrac{\varepsilon}{4}|\eta| \le \max(1,|\xi+\eta|) \le \varepsilon|\eta|\,\}. \tag{4.9}$$

In particular this yields that $\hat{e}_\varepsilon = 0$ when $1+|\xi+\eta| < |\eta|\varepsilon/4$, so e_ε fulfils (2.25) for $B = 4/\varepsilon$. Hence the terms $e_{2^{-\nu}}$ in (4.7) do so for $B = 2^{\nu+2}$.

The next result characterises the $a \in S^d_{1,1}$ for which the adjoint symbol a^* is again in $S^d_{1,1}$; cf the below condition (i). Since adjoining is an involution, such symbols constitute the class

$$\tilde{S}^d_{1,1} := S^d_{1,1} \cap (S^d_{1,1})^*. \tag{4.10}$$

Theorem 4.3. *For a symbol $a(x,\eta)$ in $S^d_{1,1}(\mathbb{R}^n\times\mathbb{R}^n)$ the following properties are equivalent:*

(i) *The adjoint symbol $a^*(x,\eta)$ is also in $S^d_{1,1}(\mathbb{R}^n\times\mathbb{R}^n)$.*

(ii) *For arbitrary $N>0$ and α, β there is a constant $C_{\alpha,\beta,N}$ such that*

$$|D^\alpha_\eta D^\beta_x a_{\chi,\varepsilon}(x,\eta)| \le C_{\alpha,\beta,N}\varepsilon^N(1+|\eta|)^{d-|\alpha|+|\beta|} \quad \text{for} \quad 0<\varepsilon<1. \tag{4.11}$$

(iii) *For all $\sigma\in\mathbb{R}$ there is a constant $c_{\alpha,\sigma}$ such that for $0<\varepsilon<1$*

$$\sup_{R>0,\ x\in\mathbb{R}^n} R^{|\alpha|-d}\Big(\int_{R\le|\eta|\le 2R} |D^\alpha_\eta a_{\chi,\varepsilon}(x,\eta)|^2\,\frac{d\eta}{R^n}\Big)^{1/2} \le c_{\alpha,\sigma}\varepsilon^{\sigma+\frac{n}{2}-|\alpha|}. \tag{4.12}$$

In the affirmative case $a\in\tilde{S}^d_{1,1}$, and there is an estimate

$$|D^\alpha_\eta D^\beta_x a^*(x,\eta)| \le (C_{\alpha,\beta}(a)+C'_{\alpha,\beta,N})(1+|\eta|)^{d-|\alpha|+|\beta|} \tag{4.13}$$

for a certain continuous seminorm $C_{\alpha,\beta}$ on $S^d_{1,1}(\mathbb{R}^n\times\mathbb{R}^n)$ and a finite sum $C'_{\alpha,\beta,N}$ of constants fulfilling the inequalities in (ii).

It should be observed that (i) holds for $a(x,\eta)$ if and only if it holds for $a^*(x,\eta)$ (neither (ii) nor (iii) make this obvious). But (ii) immediately gives the (expected) inclusion $\tilde{S}^d_{1,1}\subset\tilde{S}^{d'}_{1,1}$ for $d'>d$. Condition (iii) is close in spirit to the Mihlin–Hörmander multiplier theorem and is useful for estimates, as shown later in Section 6.

Remark 4.4. Conditions (ii), (iii) both hold either for all χ satisfying (2.31) or for none, for (i) does not depend on χ. It suffices to verify (ii) or (iii) for $0<\varepsilon<\varepsilon_0$ for some convenient $\varepsilon_0\in\,]0,1[$. This is implied Lemma 2.10 since every power ε^p is bounded on the interval $[\varepsilon_0,1]$.

The theorem was undoubtedly known to Hörmander, who stated the equivalence of (i) and (ii) explicitly in [Hör88, Thm. 4.2] and [Hör97, Thm. 9.4.2], in the latter with brief remarks on (iii). Equivalence with continuous extensions $H^{s+d}\to H^s$ for all $s\in\mathbb{R}$ was also shown.

However, the expositions there left a considerable burden of verification to the reader, and especially since a decisive corollary to the proof will follow further below, complete details should be in order here:

4.2.1. *Equivalence of* (ii) *and* (iii). That (ii) implies (iii) is seen at once by insertion, taking $\beta=0$ and $N=\sigma+\frac{n}{2}-|\alpha|$.

Conversely, note first that $|\xi+\eta|\le\varepsilon|\eta|$ in the spectrum of $a_{\chi,\varepsilon}(\cdot,\eta)$. That is, $|\xi|\le(1+\varepsilon)|\eta|$ so Bernstein's inequality gives

$$|D^\beta_x D^\alpha_\eta a_{\chi,\varepsilon}(x,\eta)| \le ((1+\varepsilon)|\eta|)^{|\beta|}\sup_{x\in\mathbb{R}}|D^\alpha_\eta a_{\chi,\varepsilon}(x,\eta)|. \tag{4.14}$$

Hence $C_{\alpha,\beta,N}=2^{|\beta|}C_{\alpha,0,N}$ is possible, so it suffices to prove (iii) $\Longrightarrow$ (ii) only for $\beta=0$.

For the corona $1\le|\zeta|\le 2$ Sobolev's lemma gives for $f\in C^\infty(\mathbb{R}^n)$,

$$|f(\zeta)|\le c_1(\sum_{|\beta|\le[n/2]+1}\int_{1\le|\zeta|\le2}|D^\beta f(\zeta)|^2\,d\zeta)^{1/2}. \tag{4.15}$$

Insertion of $D^\alpha_\eta a_{\chi,\varepsilon}(x,R\zeta)$ and substituting $\zeta=\eta/R$, whereby $R\le|\eta|\le 2R$, $R>0$,

$$\begin{aligned}|D^\alpha_\eta a_{\chi,\varepsilon}(x,\eta)| &\le c_1(\sum_{|\beta|\le[n/2]+1}R^{2|\beta|}\int_{R\le|\eta|\le2R}|D^{\alpha+\beta}_\eta a_{\chi,\varepsilon}(x,\eta)|^2\frac{d\eta}{R^n})^{1/2}\\ &\le c_1(\sum_{|\beta|\le[n/2]+1}R^{2d-2|\alpha|}C^2_{\alpha+\beta,\sigma}\varepsilon^{2(\sigma+\frac{n}{2}-|\alpha|-|\beta|)})^{1/2}\\ &\le c_1(\sum_{|\beta|\le[n/2]+1}C^2_{\alpha+\beta,\sigma})^{1/2}\varepsilon^{\sigma-1-|\alpha|}R^{d-|\alpha|}.\end{aligned} \tag{4.16}$$

Here $R^{d-|\alpha|}\le(1+|\eta|)^{d-|\alpha|}$ for $d\ge|\alpha|$, that leads to (ii) as $\sigma\in\mathbb{R}$ can be arbitrary.

For $|\alpha|>d$ it is first noted that by the support condition on χ, clearly $a_{\chi,\varepsilon}(x,\eta)\ne0$ only for $2R\ge|\eta|\ge\varepsilon^{-1}>1$. But $R\ge1/2$ yields $R^{d-|\alpha|}\le(\frac13(\frac12+2R))^{d-|\alpha|}\le6^{|\alpha|-d}(1+|\eta|)^{d-|\alpha|}$, so (ii) follows from the above.

4.2.2. *The implication* (ii) $\Longrightarrow$ (i) *and the estimate.* The condition (ii) is exploited for each term in the decomposition (4.7). Setting $b_\nu(x,\eta)=e^*_{2^{-\nu}}(x,\eta)$ it follows from Lemma 4.1 that b_ν is in $S^d_{1,1}$ by the remarks after (4.9), cf (4.7) ff, and (4.2) gives

$$|D^\alpha_\eta D^\beta_x b_\nu(x,\eta)|\le C_{\alpha,\beta}(e_\nu)2^{\nu+2}(1+2^{(\nu+2)(d-|\alpha|+|\beta|)})(1+|\eta|)^{d-|\alpha|+|\beta|}. \tag{4.17}$$

Now (ii) implies that $C_{\alpha,\beta}(a_{\chi,2^{-\nu}})\le C'_{\alpha,\beta,N}2^{-\nu N}$ for all $N>0$ (with other contants $C'_{\alpha,\beta,N}$ as the seminorms $C_{\alpha,\beta}$ may contain derivatives of higher order than $|\alpha|$ and $|\beta|$). Hence $C_{\alpha,\beta}(e_{2^{-\nu}})\le C'_{\alpha,\beta,N}2^{1-\nu N}$. It follows from this that $\sum b_\nu$ converges to some b in $S^d_{1,1}$ (in the Fréchet topology of this space), so that $a^*(x,\eta)=b(x,\eta)$ is in $S^d_{1,1}$.

More precisely, (4.2) and the above yields for $N=2+(d-|\alpha|+|\beta|)_+$

$$\begin{aligned}\frac{|D^\alpha_\eta D^\beta_x a^*(x,\eta)|}{(1+|\eta|)^{d-|\alpha|+|\beta|}} &\le 2^N C_{\alpha,\beta}(a-a_{\chi,1})+\sum_{\nu=0}^\infty C_{\alpha,\beta}(e_{2^{-\nu}})2^{\nu+2}(1+2^{(\nu+2)(d-|\alpha|+|\beta|)_+})\\ &\le 2^N C_{\alpha,\beta}(a-a_{\chi,1})+\sum_{\nu=0}^\infty 16C'_{\alpha,\beta,N}2^{-\nu(N-1)}2^{(\nu+2)(d-|\alpha|+|\beta|)_+}\\ &\le 2^N C_{\alpha,\beta}(a-a_{\chi,1})+4^{N+2}C'_{\alpha,\beta,N}.\end{aligned} \tag{4.18}$$

Invoking the continuity from Lemma 2.10 in the first term, the last statement follows.

4.2.3. *Verification of* (i) $\Longrightarrow$ (ii). It suffices to derive another decomposition

$$a = A + \sum_{\nu=0}^{\infty} a_\nu, \tag{4.19}$$

in which $A \in S^{-\infty}$ and each $a_\nu \in S^d_{1,1}$ with $\hat{a}_\nu(\xi,\eta) = 0$ for $2^{\nu+1}|\xi+\eta| < |\xi|$ and seminorms $C_{\alpha,\beta}(a_\nu) = \mathscr{O}(2^{-\nu N})$ for each $N > 0$.

Indeed, when $\chi(\xi+\eta,\varepsilon\eta) \neq 0$ the triangle inequality gives $|\xi+\eta| \le \varepsilon|\eta| \le \varepsilon|\xi+\eta| + \varepsilon|\xi|$, whence $|\xi+\eta|(1-\varepsilon)/\varepsilon \le |\xi|$, so that for one thing

$$\hat{a}_{\chi,\varepsilon}(x,\eta) = \chi(\xi+\eta,\varepsilon\eta)\hat{A}(x,\eta) + \sum_{2^{\nu+1} > (1-\varepsilon)/\varepsilon} \chi(\xi+\eta,\varepsilon\eta)\hat{a}_\nu(x,\eta) \tag{4.20}$$

Secondly, for each seminorm $C_{\alpha,\beta}$ in $S^d_{1,1}$ one has $C_{\alpha,\beta}(a_{\nu,\chi,\varepsilon}) \le \varepsilon^{-|\alpha|}C_{\alpha,\beta}(a_\nu)$ by Lemma 2.10, so by estimating the geometric series by its first term, the above formula entails that

$$C_{\alpha,\beta}(a_{\chi,\varepsilon}) \le C_{\alpha,\beta}(A_{\chi,\varepsilon}) + \sum_{2^{\nu+1} > (1-\varepsilon)/\varepsilon} \frac{C_{N+|\alpha|}}{\varepsilon^{|\alpha|}} 2^{-\nu(N+|\alpha|)} \le C_{\alpha,\beta}(A_{\chi,\varepsilon}) + c\varepsilon^{-|\alpha|}(\frac{2\varepsilon}{1-\varepsilon})^{N+|\alpha|}. \tag{4.21}$$

This gives the factor ε^N in (ii) for $0 < \varepsilon \le 1/2$. For $1/2 < \varepsilon < 1$ the series is $\mathscr{O}(\varepsilon^{-|\alpha|})$ because $2^{-\nu} \le 1 < 2\varepsilon/(1-\varepsilon)$ for all ν. But $1 \le (2\varepsilon)^{N+|\alpha|}$ for such ε, so (ii) will follow for all $\varepsilon \in]0,1[$. (It is seen directly that $|A_{\chi,\varepsilon}(x,\eta)| \le c\varepsilon^N(1+|\eta|)^d$ etc, for only the case $\varepsilon|\eta| \ge 1$ is non-trivial, and then $\varepsilon^{-N} \le (1+|\eta|)^N$ while $A \in S^{-\infty}$.)

In the deduction of (4.19) one can use a Littlewood–Paley partition of unity, say $1 = \sum_{\nu=0}^\infty \Phi_\nu$ with dilated functions $\Phi_\nu(\eta) = \Phi(2^{-\nu}\eta) \neq 0$ only for $\frac{11}{20}2^\nu \le |\eta| \le \frac{13}{10}2^\nu$ if $\nu \ge 1$. Beginning with a trivial split $a^* = A_0 + A_1$ into two terms for which $A_0 \in S^{-\infty}$ and $A_1 \in S^d_{1,1}$ such that $A_1(x,\eta) = 0$ for $|\eta| < 1/2$, this gives

$$\hat{a}^*(\xi,\eta) = \hat{A}_0(\xi,\eta) + \sum_{\nu=0}^{\infty} \Phi_\nu(\xi/|\eta|)\hat{A}_1(\xi,\eta). \tag{4.22}$$

This yields the desired $a_\nu(x,\eta)$ as the adjoint symbol to $\mathscr{F}^{-1}_{\xi\to x}\Phi_\nu(\xi/|\eta|)\hat{A}_1(\xi,\eta)$, that is to $\int |2^\nu\eta|^n \check{\Phi}(|2^\nu\eta|y)A_1(x-y,\eta)\,dy$. Indeed, it follows directly from [Hör88, Prop. 3.3] (where the proof uses Taylor expansion and vanishing moments of $\check{\Phi}$ for $\nu \ge 1$) that a^*_ν belongs to $S^d_{1,1}$ with $(2^{N\nu}a^*_\nu)_{\nu\in\mathbb{N}}$ bounded in $S^d_{1,1}$ for all $N > 0$. Therefore (4.22) gives (4.19) by inverse Fourier transformation. Moreover, since $\hat{a}^*_\nu(\xi,\eta)$ for $\nu \ge 1$ is supported by the region

$$\tfrac{11}{20}2^\nu|\eta| \le |\xi| \le \tfrac{13}{20}2^\nu|\eta|, \tag{4.23}$$

where a fortiori

$$1 + |\xi+\eta| \ge |\xi| - |\eta| \ge (\tfrac{11}{20}2^\nu - 1)|\eta| \ge \tfrac{1}{10}|\eta|, \tag{4.24}$$

it is clear that $\hat{a}^*_\nu(\xi,\eta)$ vanishes if $10(|\xi+\eta|+1)<|\eta|$. According to Lemma 4.1 this implies that $a_\nu = a^{**}_\nu$ is also in $S^d_{1,1}$ and that, because of the above boundedness in $S^d_{1,1}$,

$$|D^\alpha_\eta D^\beta_x a_\nu(x,\eta)| \le C_{\alpha,\beta}(a^*_\nu)10(1+10^{d-|\alpha|+|\beta|})(1+|\eta|)^{d-|\alpha|+|\beta|} \le c2^{-N\nu}(1+|\eta|)^{d-|\alpha|+|\beta|} \tag{4.25}$$

for some constant independent of ν. Therefore the a_ν tend rapidly to 0, which completes the proof of Theorem 4.3.

4.2.4. *Consequences for the class* $\tilde{S}^d_{1,1}$. One can set Theorem 4.3 in relation to the definition of $a(x,D)$ by vanishing frequency modulation, simply by elaborating on the above proof:

Corollary 4.5. *On* $\tilde{S}^d_{1,1}(\mathbb{R}^n\times\mathbb{R}^n)$ *the adjoint operation is stable with respect to vanishing frequency modulation in the sense that, when* $a\in\tilde{S}^d_{1,1}$, $\psi\in C^\infty_0(\mathbb{R}^n)$ *with* $\psi=1$ *around* 0, *then*

$$\big(\psi(2^{-m}D_x)a(x,\eta)\psi(2^{-m}\eta)\big)^* \xrightarrow[m\to\infty]{} a(x,\eta)^* \tag{4.26}$$

holds in the topology of $S^{d+1}_{1,1}(\mathbb{R}^n\times\mathbb{R}^n)$.

Proof. For brevity $b_m(x,\eta)=\psi(2^{-m}D_x)a(x,\eta)\psi(2^{-m}\eta)$ denotes the symbol that is frequency modulated in both variables. The proof consists in insertion of $a(x,\eta)-b_m(x,\eta)$ into (4.18), where the first sum can be read as an integral with respect to the counting measure, which tends to 0 for $m\to\infty$ by majorised convergence.

Note that for each $\nu\ge 0$ in the first sum of (4.18) one must control $C_{\alpha,\beta}(e^m_{2^{-\nu}})$ for $m\to\infty$ when $e^m_{2^{-\nu}}$ is given by

$$\hat{e}^m_{2^{-\nu}}(\xi,\eta) = (\chi(\xi+\eta,2^{-\nu}\eta)-\chi(\xi+\eta,2^{-\nu-1}\eta))(1-\psi(2^{-m}\xi)\psi(2^{-m}\eta))\hat{a}(\xi,\eta). \tag{4.27}$$

To do so, note first that a convolution estimate gives $p_{\alpha,\beta}(b_m)\le c\sum_{\gamma\le\alpha}p_{\gamma,\beta}(a)$, whence $(b_m)_{m\in\mathbb{N}}$ is bounded in $S^d_{1,1}$. Similar arguments yield that $b_m\to a$ in $S^{d+1}_{1,1}$ for $m\to\infty$; cf [Joh08b, Lem. 2.1]. Moreover, for each $\nu\ge 0$, every seminorm $p_{\alpha,\beta}$ now on $S^{d+1}_{1,1}$, gives

$$p_{\alpha,\beta}(e^m_{2^{-\nu}}) \le p_{\alpha,\beta}((a-b_m)_{\chi,2^{-\nu}}) + p_{\alpha,\beta}((a-b_m)_{\chi,2^{-\nu-1}}). \tag{4.28}$$

Here both terms on the right-hand side tend to 0 for $m\to\infty$, in view of the continuity of $a\mapsto a_{\chi,\varepsilon}$ on $S^{d+1}_{1,1}$; cf Lemma 2.10. Hence $C_{\alpha,\beta}(e^m_{2^{-\nu}})\to 0$ for $m\to\infty$.

It therefore suffices to replace d by $d+1$ in (4.18) and majorise. However, $a\mapsto a_{\chi,\varepsilon}$ commutes with $a\mapsto b_m$ as maps in $\mathscr{S}'(\mathbb{R}^n\times\mathbb{R}^n)$, so since $a\in\tilde{S}^{d+1}_{1,1}$, it follows from (ii) that

$$p_{\alpha,\beta}((a-b_m)_{\chi,\varepsilon}) \le p_{\alpha,\beta}(a_{\chi,\varepsilon}) + c\sum_{\gamma\le\alpha}p_{\gamma,\beta}(a_{\chi,\varepsilon}) \le (1+c)(\sum_{\gamma\le\alpha}C_{\gamma,\beta,N})\varepsilon^N \le C'_{\alpha,\beta,N}\varepsilon^N. \tag{4.29}$$

Using this in the previous inequality, $C_{\alpha,\beta}(e^m_{2^{-\nu}})\le C2^{-\nu N}$ is obtained for C independent of $m\in\mathbb{N}$. Now it follows from (4.18) that $b_m(x,\eta)^*\to a(x,\eta)^*$ in $S^{d+1}_{1,1}$ as desired. □

Thus prepared, the proof of Proposition 4.2 can now be repeated from (4.5) onwards, which immediately gives the first main result of the paper:

Theorem 4.6. *When a symbol $a(x,\eta)$ of type $1,1$ belongs to the class $\tilde{S}^d_{1,1}(\mathbb{R}^n\times\mathbb{R}^n)$, as characterised in Theorem 4.3, then*

$$a(x,D)\colon \mathscr{S}'(\mathbb{R}^n)\to\mathscr{S}'(\mathbb{R}^n) \tag{4.30}$$

is everywhere defined and continuous, and it equals the adjoint of $\operatorname{OP}(e^{\mathrm{i}D_x\cdot D_\eta}\bar{a}(x,\eta))$.

5. Dyadic Corona Decompositions

This section describes how Littlewood–Paley techniques provide a convenient passage to auxiliary operators, that may be analysed with pointwise estimates.

5.1. **Paradifferential splitting of symbols and operators.** Recalling the definition of type $1,1$-operators in (2.4)–(2.5), it is noted that to each modulation function ψ, ie $\psi\in C_0^\infty(\mathbb{R}^n)$ with $\psi=1$ in a neighbourhood of 0, there exist $R>r>0$ with $R\ge 1$ satisfying

$$\psi(\xi)=1 \quad\text{for}\quad |\xi|\le r; \qquad \psi(\xi)=0 \quad\text{for}\quad |\xi|\ge R. \tag{5.1}$$

For fixed ψ it is convenient to take an integer $h\ge 2$ so large that $2R<r2^h$.

To obtain a Littlewood–Paley decomposition from ψ, set $\varphi=\psi-\psi(2\cdot)$. Then a dilation of this function is supported in a corona,

$$\operatorname{supp}\varphi(2^{-k}\cdot)\subset\bigl\{\,\xi\bigm| r2^{k-1}\le|\xi|\le R2^k\,\bigr\},\qquad \text{for } k\ge 1. \tag{5.2}$$

The identity $1=\psi(x)+\sum_{k=1}^\infty\varphi(2^{-k}\xi)$ follows by letting $m\to\infty$ in the telescopic sum,

$$\psi(2^{-m}\xi)=\psi(\xi)+\varphi(\xi/2)+\dots+\varphi(\xi/2^m). \tag{5.3}$$

Using this, functions $u(x)$ and symbols $a(x,\eta)$ will be localised to frequencies $|\eta|\approx 2^j$ as

$$u_j=\varphi(2^{-j}D)u,\qquad a_j(x,\eta)=\varphi(2^{-j}D_x)a(x,\eta)=\mathscr{F}^{-1}_{\xi\to x}(\varphi(2^{-j}\xi)\hat{a}(\xi,\eta)). \tag{5.4}$$

Localisation to balls given by $|\eta|\le R2^j$ are written with upper indices,

$$u^j=\psi(2^{-j}D)u,\qquad a^j(x,\eta)=\psi(2^{-j}D_x)a(x,\eta)=\mathscr{F}^{-1}_{\xi\to x}(\psi(2^{-j}\xi)\hat{a}(\xi,\eta)). \tag{5.5}$$

In addition $u_0=u^0$ and $a_0=a^0$; by convention they are all taken to equal 0 for $j<0$. (To avoid having two different meanings of sub- and superscripts, the dilations $\psi(2^{-j}\cdot)$ are written as such, with the corresponding Fourier multiplier as $\psi(2^{-j}D)$, and similarly for φ). Note that $a^k(x,D)=\operatorname{OP}(\psi(2^{-k}D_x)a(x,\eta))$ etc.

Inserting the relation (5.3) twice in (2.4), bilinearity gives

$$\operatorname{OP}(\psi(2^{-m}D_x)a(x,\eta)\psi(2^{-m}\eta))u=\sum_{j,k=0}^{m}a_j(x,D)u_k. \tag{5.6}$$

Of course the sum may be split in three groups in which $j\le k-h$, $|j-k|<h$ and $k\le j-h$, respectively. For $m\to\infty$ this yields the paradifferential decomposition

$$a_\psi(x,D)u=a^{(1)}_\psi(x,D)u+a^{(2)}_\psi(x,D)u+a^{(3)}_\psi(x,D)u, \tag{5.7}$$

whenever a and u fit together such that the three series below converge in $\mathscr{D}'(\mathbb{R}^n)$:

$$a^{(1)}_\psi(x,D)u = \sum_{k=h}^{\infty} \sum_{j\le k-h} a_j(x,D)u_k = \sum_{k=h}^{\infty} a^{k-h}(x,D)u_k \tag{5.8}$$

$$\begin{aligned} a^{(2)}_\psi(x,D)u = \sum_{k=0}^{\infty} \big(&a_{k-h+1}(x,D)u_k + \dots + a_{k-1}(x,D)u_k + a_k(x,D)u_k \\ &+ a_k(x,D)u_{k-1} + \dots + a_k(x,D)u_{k-h+1}\big) \end{aligned} \tag{5.9}$$

$$a^{(3)}_\psi(x,D)u = \sum_{j=h}^{\infty} \sum_{k\le j-h} a_j(x,D)u_k = \sum_{j=h}^{\infty} a_j(x,D)u^{j-h}. \tag{5.10}$$

Note the convenient shorthand $a^{k-h}(x,D)$ for $\sum_{j\le k-h} a_j(x,D) = \operatorname{OP}(\psi(2^{h-k}D_x)a(x,\eta))$ etc. In this way (5.9) also has a brief form, namely

$$a^{(2)}_\psi(x,D)u = \sum_{k=0}^{\infty} ((a^k - a^{k-h})(x,D)u_k + a_k(x,D)(u^{k-1} - u^{k-h})). \tag{5.11}$$

In the following the subscript ψ is usually dropped because this auxiliary function will be fixed (ψ was left out already in a_j and a^j; cf (5.4)–(5.5)).

Remark 5.1. It was tacitly used in (5.6) and (5.8)–(5.10) that one has

$$a_j(x,D)u_k = \operatorname{OP}(a_j(x,\eta)\varphi(2^{-k}\eta))u. \tag{5.12}$$

This is because, with $\chi \in C_0^\infty$ equalling 1 on $\operatorname{supp} \mathscr{F} u_k$, both sides are equal to

$$\operatorname{OP}(a_j(x,\eta)\chi(\eta))u_k. \tag{5.13}$$

Indeed, while this is trivial for the right-hand side of (5.12), where the symbol is in $S^{-\infty}$ and $\chi \equiv 1$ on $\operatorname{supp}\varphi_k$, it is for the type $1,1$-operator on the left-hand side a fact that follows at once from (2.10) (as observed in [Joh08b]). Therefore the preliminary extension to $\mathscr{F}^{-1}\mathscr{E}'$ in (2.10) is crucial for the simple formulae in the present paper.

Analogously Definition 2.1 may be rewritten as $a(x,D)u = \lim a^m(x,D)u^m$.

The importance of the decomposition in (5.8)–(5.10) lies in the fact that the summands have their spectra in balls and coronas:

Proposition 5.2. *If $a \in S^d_{1,1}(\mathbb{R}^n \times \mathbb{R}^n)$ and $u \in \mathscr{S}'(\mathbb{R}^n)$, and r, R are chosen as in* (5.1) *for each auxiliary function ψ, then every $h \in \mathbb{N}$ such that $2R < r2^h$ gives*

$$\operatorname{supp} \mathscr{F}(a^{k-h}(x,D)u_k) \subset \big\{\, \xi \;\big|\; R_h 2^k \le |\xi| \le \frac{5R}{4} 2^k \,\big\} \tag{5.14}$$

$$\operatorname{supp} \mathscr{F}(a_k(x,D)u^{k-h}) \subset \big\{\, \xi \;\big|\; R_h 2^k \le |\xi| \le \frac{5R}{4} 2^k \,\big\}, \tag{5.15}$$

where $R_h = \frac{r}{2} - R2^{-h} > 0$.

Proof. Since $u_k \in \mathscr{F}^{-1}\mathscr{E}'$ is in the domain of the type $1,1$-operator $a^{k-h}(x,D)$, the last part of Theorem B.1 and (5.2) give

$$\operatorname{supp} \mathscr{F}(a^{k-h}(x,D)u_k) \subset \big\{\, \xi + \eta \;\big|\; (\xi,\eta) \in \operatorname{supp}(\psi_{h-k} \otimes 1)\hat{a},\ r2^{k-1} \le |\eta| \le R2^k \,\big\}. \tag{5.16}$$

So by the triangle inequality every $\zeta = \xi + \eta$ in the support fulfils

$$r2^{k-1} - R2^{k-h} \le |\zeta| \le R2^{k-h} + R2^k \le \tfrac{5}{4}R2^k, \tag{5.17}$$

as $h \ge 2$. This shows (5.14) and (5.15) follows analogously. □

To achieve less complicated constants one could take h so large that $4R \le r2^h$ instead, which would allow $R_h = r/4$ (and $9R/8$). However, the present constants are preferred in order to reduce the number of terms in $a^{(2)}(x,D)u$, as has been common in the literature.

In comparison the terms in $a^{(2)}(x,D)u$ only satisfy a dyadic ball condition. This was eg observed in [Joh05], as was the fact that when the twisted diagonal condition (2.25) can be shown to hold, then the situation improves for large k:

Proposition 5.3. *When $a \in S^d_{1,1}(\mathbb{R}^n \times \mathbb{R}^n)$, $u \in \mathscr{S}'(\mathbb{R}^n)$, and r, R are chosen as in* (5.1) *for each auxiliary function ψ, then every $h \in \mathbb{N}$ such that $2R \le r2^h$ gives*

$$\operatorname{supp}\mathscr{F}\left(a_k(x,D)(u^{k-1} - u^{k-h}) + (a^k - a^{k-h})(x,D)u_k\right) \subset \left\{\, \xi \;\middle|\; |\xi| \le 2R2^k \,\right\} \tag{5.18}$$

If $a(x,\eta)$ satisfies (2.25) *for some $B \ge 1$, the support is contained in the annulus*

$$\left\{\, \xi \;\middle|\; \frac{r}{2^{h+1}B}2^k \le |\xi| \le 2R2^k \,\right\} \tag{5.19}$$

for all $k \ge h + 1 + \log_2(B/r)$.

Proof. As in Proposition 5.2 it is seen that $\operatorname{supp}\mathscr{F} a_k(x,D)(u^{k-1} - u^{k-h})$ is contained in

$$\left\{\, \xi + \eta \;\middle|\; (\xi,\eta) \in \operatorname{supp}(\varphi_k \otimes 1)\hat{a},\ r2^{k-h} \le |\eta| \le R2^{k-1} \,\right\}. \tag{5.20}$$

Therefore any ζ in the support fulfils $|\zeta| \le R2^k + R2^{k-1} = (3R/2)2^k$. If (2.25) holds then $B(1 + |\xi + \eta|) \ge |\eta|$ on $\operatorname{supp}\mathscr{F}_{x\to\xi}a$ so that, for all k larger than the given limit,

$$|\zeta| \ge \tfrac{1}{B}|\eta| - 1 \ge \tfrac{1}{B}r2^{k-h} - 1 \ge (\tfrac{r}{2^h B} - 2^{-k})2^k \ge \tfrac{r}{2^{h+1}B}2^k. \tag{5.21}$$

The term $(a^k - a^{k-h})(x,D)u_k$ is analogous, but causes $3R/2$ to be replaced by $2R$. □

Remark 5.4. The dyadic ball and corona properties given in Proposition 5.2–5.3 have been a main reason for the introduction the paradifferential splitting (5.7) in the 1980's. However, the above inclusions were then derived under the additional assumption that $a(x,\eta)$ should be an elementary symbol; cf [Bou83, Bou88a, Yam86]. With the spectral support rule recalled in Appendix B, this is redundant. Cf also the remarks in the introduction.

5.2. **Calculation of symbols and remainder terms.** Although (5.8)–(5.10) yield a well-known splitting, the operator notation $a^{(j)}(x,D)$ requires justification in case of type $1,1$-operators.

Departing from the right hand sides of (5.8)–(5.10) one is via (5.12) at once led to the symbols

$$a^{(1)}(x,\eta) = \sum_{k=h}^{\infty} a^{k-h}(x,\eta)\varphi(2^{-k}\eta) \tag{5.22}$$

$$\begin{aligned} a^{(2)}(x,\eta) &= \sum_{k=0}^{\infty} \big((a_{k-h+1}(x,\eta)+\dots+a_{k-1}(x,\eta)+a_k(x,\eta))\varphi(2^{-k}\eta) \\ &\qquad + a_k(x,\eta)(\varphi(2^{-(k-1)}\eta)+\dots+\varphi(2^{-(k-h+1)}\eta)\big) \end{aligned} \tag{5.23}$$

$$= \sum_{k=0}^{\infty} ((a^k(x,\eta)-a^{k-h}(x,\eta))\varphi(2^{-k}\eta)+a_k(x,\eta)(\psi(2^{-(k-1)}\eta)-\psi(2^{-(k-h)}\eta))) \tag{5.24}$$

$$a^{(3)}(x,\eta) = \sum_{j=h}^{\infty} a_j(x,\eta)\psi(2^{-(j-h)}\eta). \tag{5.25}$$

These series converge in the Fréchet space $S^{d+1}_{1,1}(\mathbb{R}^n\times\mathbb{R}^n)$, for the sums are locally finite.

Actually Proposition 5.2 is closely related to the behaviour of $a^{(1)}(x,\eta)$ and $a^{(3)}(x,\eta)$ at the twisted diagonal:

Proposition 5.5. *For each $a(x,\eta)$ in $S^d_{1,1}(\mathbb{R}^n\times\mathbb{R}^n)$ and modulation function $\psi\in C_0^\infty(\mathbb{R}^n)$ as in* (5.1) *the associated symbols $a^{(1)}_\psi(x,\eta)$ and $a^{(3)}_\psi(x,\eta)$ fulfil the twisted diagonal condition* (2.25) *with constants $B_1 = 2^h(\frac{2^h r}{2R}-1)^{-1} > 2$, respectively $B_3 = (\frac{2^h r}{2R}-1)^{-1} > 1$.*

Proof. For each term in $\hat{a}^{(1)}(\xi,\eta)$ that is non-zero at (ξ,η) one has

$$|\xi+\eta| \ge 2^k(\tfrac{r}{2}-R2^{-h}) \ge |\eta|(\tfrac{r}{2R}-2^{-h}). \tag{5.26}$$

Hence $\hat{a}^{(1)}(\xi,\eta)=0$ whenever $B_1|\xi+\eta|<|\eta|$. As $B_1^{-1}<r/(2R)$, one has (2.25) for $B_1>2$. For $a^{(3)}(x,\eta)$ the corresponding calculation is $|\xi+\eta| \ge \frac{r}{2}2^j - R2^j \ge |\eta|(\frac{r2^h}{2R}-1)$. $\square$

Clearly it is natural to verify that the type $1,1$-operators corresponding to (5.22)–(5.25) are in fact given by the infinite series in (5.8)–(5.10), in particular that the series for $a^{(j)}(x,D)u$ converges precisely when u belongs to the domain of the operator $a^{(j)}(x,D)$.

In view of the definition by vanishing frequency modulation in (2.4) ff, this will necessarily be lengthy because a second auxiliary function has to be introduced.

To indicate the details for $a^{(1)}(x,\eta)$, let $\psi,\Psi\in C_0^\infty(\mathbb{R}^n)$ be equal to 1 around the origin, and let Ψ be used as the fixed auxiliary function entering $a^{(1)}(x,D)=a^{(1)}_\Psi(x,D)$ etc; and set $\Phi=\Psi-\Psi(2\cdot)$. The numbers r,R and h are then chosen in relation to Ψ as in (5.1). Moreover, ψ is used for the frequency modulation in (2.4). This gives the following identity in $S^d_{1,1}$, where prime indicates a finite sum,

$$\psi(2^{-m}D_x)a^{(1)}(x,\eta)\psi(2^{-m}\eta) = \sum_{k=h}^{m+\mu} a^{k-h}(x,\eta)\Phi(2^{-k}\eta) + \sum_k{}' \psi(2^{-m}D_x)a^{k-h}(x,\eta)\Phi(2^{-k}\eta)\psi(2^{-m}\eta). \quad (5.27)$$

In fact if $\lambda,\Lambda>0$ are chosen so that $\psi(\eta)=1$ for $|\eta|\le\lambda$ while $\psi=0$ for $|\eta|\ge\Lambda$, the support of $\Phi(2^{-k}\eta)$ lies in one of the level sets $\psi(2^{-m}\eta)=1$ or $\psi(2^{-m}\eta)=0$ when

$$R2^k \le \lambda 2^m \quad\text{or}\quad r2^{k-1}\ge\Lambda 2^m; \quad (5.28)$$

that is, with the exception of the k fulfilling

$$m+\log_2(\lambda/R) < k < m+1+\log_2(\Lambda/r). \quad (5.29)$$

This shows that the primed sum has at most $1+\log_2\frac{R\Lambda}{r\lambda}$ terms, independently of the modulation parameter m; and in addition that $\psi(2^{-m}\eta)$ and $\psi(2^{-m}D_x)$ disappear from the other terms as stated by taking $\mu=[\log_2(\lambda/R)]$.

Consequently the change of variables $k=m+l$ gives for $u\in\mathscr{S}'(\mathbb{R}^n)$ that

$$\operatorname{OP}(\psi(2^{-m}D_x)a^{(1)}(x,\eta)\psi(2^{-m}\eta))u = \sum_{k=h}^{m+\mu} a^{k-h}(x,D)u_k + \sum_{\mu<l<1+\log_2(\Lambda/r)}{}' \operatorname{OP}(\psi(2^{-m}D_x)\Psi(2^{h-l-m}D_x)a(x,\eta)\Phi(2^{-m-l}\eta)\psi(2^{-m}\eta))u. \quad (5.30)$$

A similar reasoning applies to $a^{(3)}(x,\eta)$. The main difference is that the possible inclusion of $\operatorname{supp}\Phi(2^{-j}\cdot)$, into the level sets where $\psi(2^{-m}\cdot)$ equals 1 or 0, in this case applies to the symbol $\psi(2^{-m}D_x)a_j(x,\eta)=\mathscr{F}^{-1}_{\xi\to x}(\psi(2^{-m}\xi)\Phi(2^{-j}\xi)\hat{a}(\xi,\eta))$. Therefore one has for the same μ,

$$\operatorname{OP}(\psi(2^{-m}D_x)a^{(3)}(x,\eta)\psi(2^{-m}\eta))u = \sum_{j=h}^{m+\mu} a_j(x,D)u^{j-h} + \sum_{\mu<l<1+\log_2(\Lambda/r)}{}' \operatorname{OP}(\psi(2^{-m}D_x)\Phi(2^{-l-m}D_x)a(x,\eta)\Psi(2^{h-m-l}\eta)\psi(2^{-m}\eta))u. \quad (5.31)$$

Treating $a^{(2)}_\Psi(x,D)$ analogously, it is not difficult to see that once again the central issue is whether $\operatorname{supp}\Phi(2^{-k}\cdot)$ is contained in the set where $\psi(2^{-m}\cdot)=1$ or $=0$. So for the same μ,

$$\begin{aligned}\operatorname{OP}(\psi(2^{-m}D_x)a^{(2)}(x,\eta)\psi(2^{-m}\eta))u &= \sum_{k=h}^{m+\mu} \big((a^k-a^{k-h})(x,D)u_k + a_k(x,D)(u^{k-1}-u^{k-h})\big)\\ &+ \sum_{\mu<l<1+\log_2(\Lambda/r)}{}' \operatorname{OP}(\psi(2^{-m}D_x)(a^{m+l}(x,\eta)-a^{m+l-h}(x,\eta))\Phi(2^{-m-l}\eta)\psi(2^{-m}\eta))u\\ &+ \sum_{\mu<l<1+\log_2(\Lambda/r)}{}' \operatorname{OP}(\psi(2^{-m}D_x)a_{m+l}(x,\eta)(\Psi(2^{1-m-l}\eta)-\Psi(2^{h-m-l}\eta))\psi(2^{-m}\eta))u \quad (5.32)\end{aligned}$$

To complete the programme introduced after Proposition 5.5, it only remains to let $m \to \infty$ in (5.30)–(5.32) and to show that the remainders given by the primed sums can be safely ignored:

Proposition 5.6. *When* $a(x,\eta)$ *is given in* $S^d_{1,1}(\mathbb{R}^n \times \mathbb{R}^n)$ *and* ψ, $\Psi \in C_0^\infty(\mathbb{R}^n)$ *equal* 1 *in neighbourhoods of the origin, then it holds for every* $u \in \mathscr{S}'(\mathbb{R}^n)$ *that each term (with* l *fixed) in the primed sums in* (5.30)–(5.31) *tend to* 0 *in* $\mathscr{S}'(\mathbb{R}^n)$ *for* $m \to \infty$.

This is valid for (5.32) *too, if* $a(x,\eta)$ *in addition fulfils the twisted diagonal condition* (2.25).

The verification of this result is postponed until Section 6.1, where pointwise estimates are used anyway.

Remark 5.7. For $a^{(2)}(x,D)u$ the vanishing (for $m \to \infty$) of the remainder terms in (5.32) is only claimed here in case (2.25) holds. This is because one has (5.19) then, whereas in general only the dyadic ball condition (5.18) is available. Later in Theorem 6.7 the vanishing will also be shown to hold under the milder condition (2.31). It is an open problem whether the primed sums in (5.32) will vanish for all $a \in S^d_{1,1}$, $u \in \mathscr{S}'$. (However, the split into infinite series in (5.8)–(5.10) can of course always be used even so.)

Remark 5.8. The decomposition in (5.7)–(5.10) can be traced back to Kumano-go and Nagase, who used a version of $a^{(1)}(x,\eta)$ defined by an integral to smooth non-regular symbols, cf [KgN78, Thm 1.1]. It was also important in the paradifferential calculus of Bony [Bon81], and has afterwards been convenient for the continuity analysis of pseudo-differential operators, as is evident from eg [Yam86, Mar91, Joh05, Lan06].

Remark 5.9. For pointwise multiplication decompositions analogous to (5.7) were used implicitly by Peetre [Pee76] and Triebel [Tri77]. Moreover, for $a = a(x)$ Definition 2.1 reduces to the product $\pi(a,u)$ introduced formally by the author in [Joh95] as

$$\pi(a,u) = \lim_{m\to\infty} a^m \cdot u^m. \tag{5.33}$$

This was extensively analysed in [Joh95], including continuity properties deduced from (5.7), that essentially amounts to a splitting of the generalised pointwise product $\pi(\cdot,\cdot)$ into paraproducts. Partial associativity was obtained in [Joh08b, Thm. 6.7], though.

6. Action of type $1,1$-operators on temperate distributions

In this section the paradifferential decomposition (5.7) is analysed using the pointwise estimates in Section 3, leading to fundamental Littlewood–Paley results for type $1,1$-operators; cf Theorems 6.3, 6.5 and 6.7 below.

6.1. **Polynomial bounds for the paradifferential splitting.** In the treatment of $a^{(1)}(x,D)u$ and $a^{(3)}(x,D)u$ in (5.8) and (5.10) one may conveniently commence by observing that, according to Proposition 5.2, the terms in these series fulfil condition (A.1) in Lemma A.1 for $\theta_0 = \theta_1 = 1$.

So to deduce their convergence from Lemma A.1, it remains to obtain the polynomial bounds in (A.2). This is a natural opportunity to use the efficacy of the pointwise estimates in [Joh10a], and Proposition 3.6 at once gives

Proposition 6.1. *If $a(x,\eta)$ is in $S^d_{1,1}(\mathbb{R}^n\times\mathbb{R}^n)$ and $N\ge \operatorname{order}_{\mathscr{S}'}(\mathscr{F}u)$, then*

$$|a^{k-h}(x,D)u_k(x)|\le c2^{k(N+d)}(1+|x|)^N, \tag{6.1}$$

$$|a_k(x,D)u^{k-h}(x)|\le c2^{k(N+d)_+}(1+|x|)^N. \tag{6.2}$$

Proof. The last claim follows by taking the two cut-off functions in Proposition 3.6 as Φ and $\Psi(2^{-h}\cdot)$, in the notation of Section 5. The first claim is seen by interchanging their roles, that is, by using $\Psi(2^{-h}\cdot)$ respectively Φ; the latter is 0 around the origin so $N+d$ is obtained without the positive part. □

The difference in the above estimates appears because u_k in (6.1) has spectrum in a corona. However, one should not confound this with spectral inclusions like (A.1) that one might obtain after application of $a^{k-h}(x,D)$, for these are irrelevant for the pointwise estimates here.

Therefore it is clear that similar estimates hold for the terms in $a^{(2)}(x,D)u$ as well. For example, taking $\Psi-\Psi(2^{-h}\cdot)$ and Φ, respectively, as the cut-off functions in Proposition 3.6, one finds the estimate of the first term below. Note that the positive part can be avoided for $0\le k\le h$ by using a sufficiently large constant.

Proposition 6.2. *If $a(x,\eta)$ is in $S^d_{1,1}(\mathbb{R}^n\times\mathbb{R}^n)$ and $N\ge \operatorname{order}_{\mathscr{S}'}(\mathscr{F}u)$, the terms in $a^{(2)}_\psi(x,D)u$ fulfil*

$$|(a^k-a^{k-h})(x,D)u_k(x)|+|a_k(x,D)(u^{k-1}-u^{k-h})(x)|\le c2^{k(N+d)}(1+|x|)^N. \tag{6.3}$$

Finally, using the full generality of Proposition 3.6 once more, one also obtains a

Proof of Proposition 5.6. To show that each remainder term tends to 0 for $m\to\infty$ and fixed l, it suffices to verify (A.1) and (A.2) in view of Remark A.2.

For $a^{(1)}_\Psi(x,D)$, note that by repeating the proof of Proposition 5.2 (and ignoring ψ) each remainder in (5.30) has ξ in its spectrum only when $(R_0 2^l)2^m\le|\xi|\le\frac{5\cdot 2^l}{4}R2^m$.

Moreover, each remainder term is $\le c2^{k(N+d)}(1+|x|)^N$ according to Proposition 3.6, for with the cut-off functions $\psi\Psi(2^{h-l}\cdot)$ and $\Phi(2^{-l}\cdot)\psi$ the latter is 0 around the origin. Hence a further crude estimate by $c2^{k(N+d_+)}(1+|x|)^{N+d_+}$ shows that (A.2) is fulfilled.

Similar arguments apply for the primed sum in (5.31), for $\Psi(2^{h-l}\cdot)\psi$ is 1 around the origin; which again results in the bound $c2^{k(N+d_+)}(1+|x|)^{N+d_+}$.

The procedure also works for (5.32), for (A.1) is verified as in Proposition 5.3, cf (5.19), because the extra spectral localisations provided by $\psi(2^{-m}\cdot)$ cannot increase the spectra. For the pointwise estimates one may now use eg $\psi\Phi(2^{-l}\cdot)$ and $(\Psi(2^{1-l}\cdot)-\Psi(2^{h-l}\cdot))\psi$ as the cut-off functions in the last part of (5.32). This yields the proof of Proposition 5.6.

6.2. **Littlewood–Paley analysis of type $1,1$-operators.** In the following result on the decomposition in (5.8)–(5.10), one should note in particular the confirmation that $a^{(2)}(x,D)$ induces no anomalies in case $a(x,\eta)$ fulfils the twisted diagonal condition (2.25): one may treat all (but finitely many) terms in $a^{(2)}(x,D)u$ in the same way as for $a^{(1)}(x,D)u$ and $a^{(3)}(x,D)u$, simply because they too fulfil the dyadic corona condition when (2.25) holds.

Theorem 6.3. *When $a(x,\eta)$ is a symbol in $S^d_{1,1}(\mathbb{R}^n\times\mathbb{R}^n)$ for some $d\in\mathbb{R}$ and $\psi\in C_0^\infty(\mathbb{R}^n)$ equals 1 around the origin, then the associated type $1,1$-operators $a^{(1)}_\psi(x,D)$ and $a^{(3)}_\psi(x,D)$ are everywhere defined continuous linear maps*

$$a^{(1)}_\psi(x,D),\ a^{(3)}_\psi(x,D)\colon \mathscr{S}'(\mathbb{R}^n)\to\mathscr{S}'(\mathbb{R}^n), \tag{6.4}$$

that are given by formulae (5.8) *and* (5.10), *where the infinite series converge rapidly in $\mathscr{S}'(\mathbb{R}^n)$ for every $u\in\mathscr{S}'(\mathbb{R}^n)$. The adjoints are also in $\mathrm{OP}(S^d_{1,1}(\mathbb{R}^n\times\mathbb{R}^n))$.*

If furthermore $a(x,\eta)$ fulfils (2.25), *these conclusions are valid verbatim for the operator $a^{(2)}_\psi(x,D)$, except that it is given by the series in* (5.9).

Proof. As the symbols $a^{(1)}_\Psi(x,\eta)$ and $a^{(3)}_\Psi(x,\eta)$ both belong to $S^d_{1,1}$ and fulfil (2.25) by Proposition 5.5, the corresponding operators are defined and continuous on $\mathscr{S}'(\mathbb{R}^n)$ by Proposition 4.2, with $a^{(1)}(x,D)^*$ and $a^{(3)}(x,D)^*$ both of type $1,1$.

As $\operatorname{supp}\mathscr{F}_{x\to\xi}a^{(2)}\subset\operatorname{supp}\mathscr{F}_{x\to\xi}a$, the preceding argument also applies to $a^{(2)}(x,D)$ when $a(x,\eta)$ satisfies (2.25).

Moreover, the series $\sum_{k=0}^\infty a^{k-h}(x,D)u_k$ in (5.8) converges rapidly in $\mathscr{S}'$ for every $u\in\mathscr{S}'$. This follows from 1° of Lemma A.1, for the terms fulfil (A.1) and (A.2) by Proposition 5.2, cf (5.14), and Proposition 6.1, respectively. (The latter gives a bound by $2^{k(N+d_+)}(1+|x|)^{N+d_+}$.)

Similarly Lemma A.1 yields convergence of the series (5.10) for $a^{(3)}(x,D)u$ when $u\in\mathscr{S}'$. In view of Proposition 5.3 and Proposition 6.2, convergence of the series for $a^{(2)}(x,D)u$ in (5.9) also follows from Lemma A.1.

To identify these series with the operators it remains to apply Proposition 5.6. □

It should be emphasized that duality methods and pointwise estimates contribute in two different ways in Theorem 6.3: once the symbol $a^{(1)}(x,\eta)$ has been introduced, continuity on $\mathscr{S}'(\mathbb{R}^n)$ of the associated type $1,1$-operator $a^{(1)}(x,D)$ is obtained by duality through Proposition 4.2. But the pointwise estimates in Section 3 yield (vanishing of the remainder terms, hence) the identification of $a^{(1)}(x,D)u$ with the series in (5.8). Furthermore, the pointwise estimates also give an explicit proof of the fact that $a^{(1)}(x,D)$ is defined on the entire $\mathscr{S}'(\mathbb{R}^n)$, for the right-hand side of (5.8) does not depend on the modulation function ψ. Similar remarks apply to $a^{(3)}(x,D)$. Thus duality methods and pointwise estimates together lead to a deeper analysis of type $1,1$-operators.

Remark 6.4. Theorem 6.3 generalises a result of Coifman and Meyer [MC97, Ch. 15] in three ways. They stated Lemma A.1 for $\theta_0=\theta_1=1$ and derived a corresponding fact for paramultiplication, though only with a treatment of the first and third term.

Changing focus back to the given operator $a(x,D)$, one can by means of (5.7) restate Theorem 6.3 as follows:

Theorem 6.5. *When $a\in S^d_{1,1}(\mathbb{R}^n\times\mathbb{R}^n)$ fulfils the twisted diagonal condition* (2.25), *then the associated type $1,1$-operator $a(x,D)$ defined by vanishing frequency modulation is an everywhere defined continuous linear map*

$$a(x,D)\colon\mathscr{S}'(\mathbb{R}^n)\to\mathscr{S}'(\mathbb{R}^n), \tag{6.5}$$

with its adjoint $a(x,D)^*$ *also in* $\operatorname{OP}(S^d_{1,1}(\mathbb{R}^n\times\mathbb{R}^n))$. *The operator fulfils*

$$a(x,D)u = a^{(1)}_\psi(x,D)u + a^{(2)}_\psi(x,D)u + a^{(3)}_\psi(x,D)u \tag{6.6}$$

for every $\psi\in C_0^\infty(\mathbb{R}^n)$ *equal to* 1 *in a neighbourhood of the origin, and the series in* (5.8), (5.9), (5.10) *converge rapidly in* $\mathscr{S}'(\mathbb{R}^n)$ *for every* $u\in\mathscr{S}'(\mathbb{R}^n)$.

For general $a(x,D)$, Theorem 6.3 at least shows that $a^{(1)}(x,D)u$ and $a^{(3)}(x,D)u$ are always defined. So, by accepting that the operator notation $a^{(2)}(x,D)u$ has not yet been justified in all cases, the theorem also gives

Corollary 6.6. *For* $a(x,\eta)$ *in* $S^d_{1,1}(\mathbb{R}^n\times\mathbb{R}^n)$ *a distribution* $u\in\mathscr{S}'(\mathbb{R}^n)$ *belongs to the domain* $D(a(x,D))$ *if and only if the series for* $a^{(2)}(x,D)u$ *in* (5.9) *converges in* $\mathscr{D}'(\mathbb{R}^n)$.

6.3. **The twisted diagonal condition of arbitrary order.** The above results will now be extended to the more general situation where $a(x,\eta)$ is in $\tilde{S}^d_{1,1}$, which by Theorem 4.3 means that a fulfils the twisted diagonal condition of arbitrary real order σ in (2.31). The estimates there enter the convergence proof for $a^{(2)}(x,D)u$ directly. The full generality with $\theta_0<\theta_1$ in the corona criterion Lemma A.1 is needed now, but does not alone suffice for this case.

Theorem 6.7. *Suppose* $a(x,\eta)$ *in* $\tilde{S}^d_{1,1}(\mathbb{R}^n\times\mathbb{R}^n)$, *ie* $a(x,\eta)$ *fulfils one of the equivalent conditions in Theorem 4.3. Then the conclusions of Theorem 6.5 are valid for* $a(x,D)$, *and furthermore the type* $1,1$*-operator* $\operatorname{OP}(a^{(2)}_\psi(x,\eta))$ *is given by the infinite series for* $a^{(2)}_\psi(x,D)u$ *in* (5.9).

Proof. The continuity on $\mathscr{S}'$ is assured by Theorem 4.6. For the convergence of the series in the paradifferential splitting, it is convenient to write, in the notation of (2.31) ff,

$$a(x,\eta) = (a(x,\eta)-a_{\chi,1}(x,\eta)) + a_{\chi,1}(x,\eta), \tag{6.7}$$

where $a-a_{\chi,1}$ satisfies (2.25) for $B=1$, so that Theorem 6.5 applies to it. As $a_{\chi,1}$ is in $\tilde{S}^d_{1,1}$ like a and $a-a_{\chi,1}$ (the latter by Proposition 4.2), one may reduce to the case in which

$$\hat{a}(x,\eta)\neq 0 \implies \max(1,|\xi+\eta|)\le|\eta|. \tag{6.8}$$

Continuing under this assumption, it is according to Corollary 6.6 enough to show for all $u\in\mathscr{S}'$ that there is convergence of the two series

$$\sum_{k=0}^{\infty}(a^k-a^{k-h})(x,D)u_k, \qquad \sum_{k=1}^{\infty}a_k(x,D)(u^{k-1}-u^{k-h}). \tag{6.9}$$

Since the distributions here are functions of polynomial growth by Proposition 6.2, it suffices to improve the estimates there; and to do so for $k\ge h$, respectively $k\ge 2$.

Using Hörmander's localisation to a neighbourhood of $\mathscr{T}$, cf (2.28)–(2.30), one arrives at

$$\hat{a}_{k,\chi,\varepsilon}(\xi,\eta) = \hat{a}(\xi,\eta)\Phi(2^{-k}\xi)\chi(\xi+\eta,\varepsilon\eta), \tag{6.10}$$

This leaves the remainder $b_k(x,\eta)=a_k(x,\eta)-a_{k,\chi,\varepsilon}(x,\eta)$, that applied to the above difference $v_k=u^{k-1}-u^{k-h}=\mathscr{F}^{-1}((\Phi(2^{1-k}\cdot)-\Phi(2^{h-k}\cdot))\hat{u})$ gives

$$a_k(x,D)v_k = a_{k,\chi,\varepsilon}(x,D)v_k + b_k(x,D)v_k. \tag{6.11}$$

To utilise the pointwise estimates, set $N = \operatorname{order}_{\mathscr{S}'}(\hat{u})$ and take $\psi \in C_0^\infty(\mathbb{R}^n)$ equal to 1 in a neighbourhood of the corona $\frac{r}{R}2^{-1-h} \le |\eta| \le 1$ and equal to 0 outside the set with $\frac{r}{R}2^{-2-h} \le |\eta| \le 2$. Taking the dilated function $\psi(\eta/(R2^k))$ as the auxiliary function in the symbol factor, the factorisation inequality (3.1) and Theorem 3.2 give

$$\begin{aligned} |a_{k,\chi,\varepsilon}(x,D)v_k(x)| &\le F_{a_{k,\chi,\varepsilon}}(N,R2^k;x)v_k^*(N,R2^k;x) \\ &\le cv_k^*(x) \sum_{|\alpha|=0}^{N+[n/2]+1} \Big(\int_{r2^{k-h-2}\le|\eta|\le R2^{k+1}} |(R2^k)^{|\alpha|-n/2} D_\eta^\alpha a_{k,\chi,\varepsilon}(x,\eta)|^2\,d\eta\Big)^{1/2}. \end{aligned} \tag{6.12}$$

Here the ratio of the limits is $2R/(r2^{-h-2}) > 32$, so with extension to $|\eta| \in [R2^{k+1-L}, R2^{k+1}]$ there is $L \ge 6$ dyadic coronas. This gives an estimate by $c(R2^k)^d L^{1/2} N_{\chi,\varepsilon,\alpha}(a_k)$. In addition, Minkowski's inequality gives

$$N_{\chi,\varepsilon,\alpha}(a_k) \le \sup_{\rho>0} \rho^{|\alpha|-d} \int_{\mathbb{R}^n} |2^{kn}\check{\Phi}(2^k y)| \Big(\int_{\rho\le|\eta|\le 2\rho} |D_\eta^\alpha a_{\chi,\varepsilon}(x-y,\eta)|^2 \frac{d\eta}{\rho^n}\Big)^{1/2} dy \le cN_{\chi,\varepsilon,\alpha}(a). \tag{6.13}$$

So it follows from the above that

$$|a_{k,\chi,\varepsilon}(x,D)v_k(x)| \le cv_k^*(N,R2^k;x) \Big(\sum_{|\alpha|\le N+[n/2]+1} c_{\alpha,\sigma} \varepsilon^{\sigma+n/2-|\alpha|} \Big) L^{1/2} (R2^k)^d. \tag{6.14}$$

Using Lemma 3.1 and taking $\varepsilon = 2^{-k\theta}$, say for $\theta = 1/2$ this gives

$$|a_{k,\chi,2^{-k\theta}}(x,D)v_k(x)| \le c(1+|x|)^N 2^{-k(\sigma-1-2d-3N)/2}. \tag{6.15}$$

Choosing $\sigma > 3N+2d+1$, the series $\sum_k \langle a_{k,\chi,\varepsilon}(x,D)v_k,\, \varphi \rangle$ converges rapidly for $\varphi \in \mathscr{S}$.

To treat $\sum_{k=0}^\infty b_k(x,D)v_k$ it is observed that $\hat{a}_{k,\chi,2^{-k\theta}}(x,\eta) = \hat{a}_k(x,\eta)$ holds in the set where $\chi(\xi+\eta, 2^{-k\theta}\eta) = 1$, that is, when $2\max(1,|\xi+\eta|) \le 2^{-k\theta}|\eta|$, so by (6.8),

$$\operatorname{supp} \hat{b}_k \subset \big\{\, (\xi,\eta) \;\big|\; 2^{-1-k\theta}|\eta| \le \max(1,|\xi+\eta|) \le |\eta| \,\big\}. \tag{6.16}$$

This implies by Theorem B.1 that $\zeta = \xi+\eta$ is in $\operatorname{supp} \mathscr{F} b_k(x,D)v_k$ only if both

$$|\zeta| \le |\eta| \le R2^k \tag{6.17}$$

$$\max(1,|\zeta|) \ge 2^{-1-k\theta}|\eta| \ge r2^{k(1-\theta)-h-2}. \tag{6.18}$$

When k fulfils $2^{k(1-\theta)} > 2^{h+2}/r$, so that the last right-hand side is > 1, these inequalities give

$$(r2^{-h-2})2^{k(1-\theta)} \le |\zeta| \le R2^k. \tag{6.19}$$

This shows that the corona condition (A.1) in Lemma A.1 is fulfilled for $\theta_0 = 1-\theta = 1/2$ and $\theta_1 = 1$, and the growth condition (A.2) is easily checked since both $a_{k,\chi,\varepsilon}(x,D)v_k$ and $a_k(x,D)v_k$ are estimated by $2^{k(N+d_+)}(1+|x|)^{N+d_+}$, as can be seen from (6.15) and Proposition 3.6, respectively. Hence $\sum b_k(x,D)v_k$ converges rapidly.

For the series $\sum_{k=0}^\infty |\langle (a^k - a^{k-h})(x,D)u_k,\, \varphi \rangle|$ it is not complicated to modify the above. Indeed, the pointwise estimates of the v_k^* are easily carried over to u_k^*, for $R2^k$ was used as the outer spectral radius of v_k; and $r2^{k-h-1}$ may also be used as the inner spectral radius of u_k. In addition the symbol $a^k - a^{k-h}$ can be treated by replacing $\Phi(2^{-k}\xi)$ by $\Psi(2^{-k}\xi) - \Psi(2^{h-k}\xi)$ in (6.10) ff., for the use of Minkowski's inequality will now give the factor $\int |\Psi - \Psi(2^h\cdot)|\,dy$ in the constant. The treatment of $b_k(x,D)v_k$ may be used verbatim for

$$\tilde{b}_k(x,D)u_k = (a^k - a^{k-h})(x,D)u_k - (a^k - a^{k-h})_{\chi,\varepsilon}(x,D)u_k. \tag{6.20}$$

Concerning the remainder terms, one can now simply carry over the above arguments to each term in the primed sums in (5.32). Indeed, for $k = m + l$ the function $\Phi(2^{-k}\xi)$ in (6.10) should then be replaced by $\psi(2^{-m}\xi)\Phi(2^{-m-l}\xi)$, whereas in v_k the Fourier multiplier should now be $\psi(2^{-m}\cdot)(\Phi(2^{1-l-m}\cdot) - \Phi(2^{h-l-m}\cdot))$. Ignoring the localisations provided by $\psi(2^{-m}\cdot)$, these changes only give other constants, so the contributions analogous to (6.10) ff. tend to 0 for a large σ, respectively by Remark A.2. The first primed sum in (5.32) can be similarly treated. □

The detailed analysis in Theorem 6.7 is exploited in the next section.

7. L_p-ESTIMATES

As another application of the paradifferential splitting (5.7), it would be natural to explain how it leads to boundedness of $a(x,D)$ in the scale of Sobolev spaces $H^s_p = \mathrm{OP}((1+|\cdot|^2)^{-s/2})L_p$.

However, because of the Littlewood–Paley analysis that will follow, it requires almost no extra effort to cover the more general Besov spaces $B^s_{p,q}$ and Lizorkin–Triebel spaces $F^s_{p,q}$. It is recalled that there are well-known identifications such as

$$H^s_p = F^s_{p,2} \quad \text{for} \quad 1 < p < \infty, \tag{7.1}$$

$$C^s_* = B^s_{\infty,\infty} \quad \text{for} \quad s \in \mathbb{R}, \tag{7.2}$$

where C^s_* denotes the Hölder–Zygmund spaces, defined eg as in [Hör97, Def. 8.6.4].

Example 7.1. In the $F^s_{p,q}$-scale, $f(t) = \sum_{j=0}^{\infty} 2^{-jd}e^{\mathrm{i}2^j t}$ belongs locally to $F^d_{p,\infty}(\mathbb{R})$; cf [Joh08b, Rem. 3.7]. This is for $0 < d \le 1$ a variant of Weierstrass' nowhere differentiable function.

Homogeneous distributions were characterised in the $B^s_{p,q}$-scale in Prop. 2.8 of [Joh08a]: when $u \in \mathscr{D}'(\mathbb{R}^n)$ is C^∞ on $\mathbb{R}^n \setminus \{0\}$ and homogeneous of degree $a \in \mathbb{C}$ there (cf [Hör85, Def 3.2.2]), then (at $x = 0$) u is locally in $B^{\frac{n}{p}+\operatorname{Re} a}_{p,\infty}(\mathbb{R}^n)$ for $0 < p \le \infty$. If $-n < \operatorname{Re} a < 0$ and $p \in\,]-\frac{n}{\operatorname{Re} a}, \infty]$ then $u \in B^{\frac{n}{p}+\operatorname{Re} a}_{p,\infty}(\mathbb{R}^n)$; this holds also for $p = \infty$ if $\operatorname{Re} a = 0$. These conclusions are optimal with respect to s and q, unless u is a homogenenous polynomial (the only case in which $u \in C^\infty(\mathbb{R}^n)$). Eg $\delta_0 \in B^{\frac{n}{p}}_{p,\infty}$ while a quotient of two homogeneous polynomials of the same degree, say $P(x)/Q(x)$ is locally in $B^{\frac{n}{p}}_{p,\infty}$ for $0 < p \le \infty$.

To invoke the $B^s_{p,q}$ and $F^s_{p,q}$ scales is natural in the context, for it was shown in [Joh04, Joh05] that every type $1,1$-operator $a(x,D)$ of order $d \in \mathbb{R}$ is a bounded map

$$a(x,D)\colon F^d_{p,1}(\mathbb{R}^n) \to L_p(\mathbb{R}^n) \quad \text{for } 1 \le p < \infty. \tag{7.3}$$

Because $B^d_{p,1} \subset F^d_{p,1}$ is a strict inclusion for $p > 1$, this sharpened the borderline analysis of Bourdaud [Bou88a]; (7.3) was moreover proved to be optimal within the $B^s_{p,q}$- and $F^s_{p,q}$-scales.

To recall the definition of $B^s_{p,q}$ and $F^s_{p,q}$, let $1=\sum_{j=0}^\infty \Phi_j(\eta)$ be a Littlewood–Paley partition of unity with $\Phi_j=\Phi(2^{-j}\cdot)$ for $\Phi=\Psi-\Psi(2\cdot)$, though $\Phi_0=\Psi$, whereby $\Psi\in C_0^\infty(\mathbb{R}^n)$ equal to 1 around the origin is fixed; cf (5.3). Usually it has been required that $\operatorname{supp}\Phi$ should be contained in the corona with $\frac12\le|\xi|\le 2$; but this restriction is avoided here in order that Ψ can be taken equal to an arbitrary modulation function entering $a(x,D)$. That this is possible can be seen by adopting the approach in eg [Yam86, JS08]:

When Ψ is fixed as above, then the spaces are defined for $s\in\mathbb{R}$ and $p,q\in\,]0,\infty]$ as follows, when $\|\cdot\|_p$ denotes the (quasi-)norm of the Lebesgue space $L_p(\mathbb{R}^n)$ for $0<p\le\infty$ and $\|\cdot\|_{\ell_q}$ stands for that of the sequence space $\ell_q(\mathbb{N}_0)$,

$$B^s_{p,q}(\mathbb{R}^n)=\big\{\,u\in\mathscr{S}'(\mathbb{R}^n)\bigm|\big\|\{2^{sj}\|\Phi_j(D)u(\cdot)\|_p\}_{j=0}^\infty\big\|_{\ell_q}<\infty\,\big\},\tag{7.4}$$

$$F^s_{p,q}(\mathbb{R}^n)=\big\{\,u\in\mathscr{S}'(\mathbb{R}^n)\bigm|\big\|\|\{2^{sj}\Phi_j(D)u\}_{j=0}^\infty\|_{\ell_q}(\cdot)\big\|_p<\infty\,\big\}.\tag{7.5}$$

Throughout it will be understood that $p<\infty$ when Lizorkin–Triebel spaces $F^s_{p,q}$ are considered.

In the definition the finite expressions are norms for $p,q\ge1$ (quasi-norms if $p<1$ or $q<1$). In general $u\mapsto\|u\|^\lambda$ is subadditive for $\lambda\le\min(1,p,q)$, so $\|f-g\|^\lambda$ is a metric.

This implies continuous embeddings $\mathscr{S}\hookrightarrow B^s_{p,q}\hookrightarrow\mathscr{S}'$ and $\mathscr{S}\hookrightarrow F^s_{p,q}\hookrightarrow\mathscr{S}'$ in the usual way, thence completeness (cf [JS07, Tri83]). There are simple embeddings $F^s_{p,q}\hookrightarrow F^{s'}_{p,r}$ for $s'<s$ and arbitrary q,r, or for $s'=s$ when $r\ge q$. Similarly for $B^s_{p,q}$.

Invoking a multiplier result, one finds a dyadic ball and corona criterion:

Lemma 7.2. *Let $s>\max(0,\frac np-n)$ for $0<p<\infty$ and $0<q\le\infty$ and suppose $u_j\in\mathscr{S}'(\mathbb{R}^n)$ such that, for some $A>0$,*

$$\operatorname{supp}\mathscr{F}u_j\subset B(0,A2^j),\qquad F(q):=\big\|(\sum_{j=0}^\infty 2^{sjq}|u_j(\cdot)|^q)^{\frac1q}\big\|_p<\infty.\tag{7.6}$$

Then $\sum_{j=0}^\infty u_j$ converges in $\mathscr{S}'(\mathbb{R}^n)$ to some $u\in F^s_{p,r}(\mathbb{R}^n)$ for $r\ge q$, $r>\frac{n}{n+s}$, and $\|u\|_{F^s_{p,r}}\le cF(r)$ for some $c>0$ depending on n, s, p and r.

When moreover $\operatorname{supp}\mathscr{F}u_j\subset\{\,\xi\mid A^{-1}2^j\le|\xi|\le A2^j\,\}$ for $j\ge J$ for some $J\ge1$, then the conclusions are valid for all $s\in\mathbb{R}$ and $r=q$.

This is an isotropic version of [JS08, Lem. 3.19-20], where the proof is applicable for arbitrary Littlewood–Paley partitions, though with other constants if Ψ is such that $R>2$. Alternatively the reader may refer to the below Proposition 7.7, where the proof also covers the sufficiency of (7.6) and in special cases gives the last part of Lemma 7.2 as well.

From Lemma 7.2 it follows that $F^s_{p,q}$ is independent of the particular Littlewood–Paley decomposition, and that different choices lead to equivalent quasi-norms.

The functions $u_k=\Phi(2^{-k}D)u$ will play a central role below because their maximal functions u_k^* are controlled in terms of the Lizorkin–Triebel norm $\|u\|_{F^s_{p,q}}$ as follows: for $0<t<\infty$ there is an estimate, cf [Yam86, Thm. 2.10], in terms of the modified Hardy-Littlewood maximal function given by $M_tu_k(x)=\sup_{r>0}(r^{-n}\int_{|x-y|\le r}|u(y)|^t\,dy)^{1/t}$,

$$u_k^*(N,R2^k;x)\le u_k^*(\tfrac nt,R2^k;x)\le cM_tu_k(x),\quad N\ge n/t.\tag{7.7}$$

So for $t < \min(p,q)$ the Fefferman-Stein inequality (cf [Yam86, Thm. 2.2]) yields a basic inequality valid for the $u_k^* = u_k^*(N, R2^k, \cdot)$ and any $s \in \mathbb{R}$,

$$\int_{\mathbb{R}^n} \|2^{sk} u_k^*(\cdot)\|_{\ell_q}^p \,dx \le c \int_{\mathbb{R}^n} \|2^{sk} M_t u_k(\cdot)\|_{\ell_q}^p \,dx \le c' \int_{\mathbb{R}^n} \|2^{sk} u_k(\cdot)\|_{\ell_q}^p \,dx = c' \|u\|_{F^s_{p,q}}^p. \tag{7.8}$$

As general references to the theory of these function spaces, the reader is referred to the books [RS96, Tri83, Tri92]; the paper [Yam86] gives a concise (anisotropic) presentation.

Remark 7.3. As an alternative to the techniques in Section 3, there is an estimate for symbols $b(x,\eta)$ in $L_{1,\mathrm{loc}}(\mathbb{R}^{2n}) \cap \mathscr{S}'(\mathbb{R}^{2n})$ with support in $\mathbb{R}^n \times \overline{B}(0,2^k)$ and $\operatorname{supp} \mathscr{F}u \subset \overline{B}(0,2^k)$, $k \in \mathbb{N}$:

$$|b(x,D)v(x)| \le c \big\|b(x,2^k\cdot)\big\|_{\dot{B}^{n/t}_{1,t}} M_t u(x), \qquad 0 < t \le 1. \tag{7.9}$$

This is Marschall's inequality, it goes back to [Mar85, p.37] and was exploited in eg [Mar91]; in the above form it was proved in [Joh05] under the condition that the right-hand side is in $L_{1,\mathrm{loc}}(\mathbb{R}^n)$ (cf also [JS08]). While $M_t u$ is as in (7.7), the homogeneous Besov norm of the symbol is of special interest here. It is defined in terms of a partition of unity $1 = \sum_{j=-\infty}^{\infty} \Phi(2^{-j}\eta)$, with Φ as in (7.4), and (7.12) read with ℓ_q over $\mathbb{Z}$ gives the norm. This yields the well-known dyadic scaling property that

$$\big\|b(x,2^k\cdot)\big\|_{\dot{B}^{n/t}_{1,t}} = 2^{k(\frac{n}{t}-n)} \big\|b(x,\cdot)\big\|_{\dot{B}^{n/t}_{1,t}}. \tag{7.10}$$

7.1. **Basic estimates in L_p.** For general type 1,1-operators $a(x,D)$ one has the next result. This appeared in [Joh05, Cor. 6.2], albeit with a rather sketchy explanation. Therefore a full proof is given here, now explicitly based on Definition 2.1 and Section 3:

Theorem 7.4. *If $a \in S^d_{1,1}(\mathbb{R}^n \times \mathbb{R}^n)$ the corresponding operator $a(x,D)$ is a bounded map for all $s > \max(0, \frac{n}{p} - n)$, $0 < p,q \le \infty$,*

$$a(x,D)\colon F^{s+d}_{p,q}(\mathbb{R}^n) \to F^s_{p,r}(\mathbb{R}^n), \quad p < \infty,\ r \ge q,\ r > n/(n+s), \tag{7.11}$$

$$a(x,D)\colon B^{s+d}_{p,q}(\mathbb{R}^n) \to B^s_{p,q}(\mathbb{R}^n). \tag{7.12}$$

Here the twisted diagonal condition (2.25) *implies* (7.11) *and* (7.12) *for all $s \in \mathbb{R}$ and $r = q$.*

Proof. Let ψ denote an arbitrary modulation function, and recall the notation from Section 5, in particular (5.7) and R, r and h. It is exploited below that $\|u\|_{F^s_{p,q}}$ can be calculated in terms of the Littlewood–Paley partition associated with ψ.

For $a^{(1)}(x,D)u = \sum_{k=h}^{\infty} a^{k-h}(x,D)u_k$ and $u \in F^s_{p,q}$ the symbol factor $F_{a^{k-h}}$ can be handled with a convolution estimate as in the proof of Proposition 3.6, so

$$|a^{k-h}(x,D)u_k(x)| \le F_{a^{k-h}}(N,R2^k;x) u_k^*(N,R2^k;x) \le c_1 \|\mathscr{F}^{-1}\psi\|_1 p(a) (R2^k)^d u_k^*(x). \tag{7.13}$$

Applying the norms of ℓ_q and L_p one has (if $q < \infty$ for simplicity's sake)

$$\int_{\mathbb{R}^n} (\sum_{k=0}^{\infty} 2^{skq} |a^{k-h}(x,D)u_k(x)|^q)^{\frac{p}{q}} \,dx \le c_2 p(a)^p \big\|(\sum_{k=0}^{\infty} 2^{(s+d)kq} u_k^*(x)^q)^{\frac{1}{q}}\big\|_p^p. \tag{7.14}$$

If $N > n/\min(p,q)$ here, it is seen from (7.8) that one has the bound in Lemma 7.2 for all $s \in \mathbb{R}$, whilst the corona condition holds by Proposition 5.2, so the lemma gives

$$\|a^{(1)}(x,D)u\|_{F^s_{p,q}} \le c\Big(\int_{\mathbb{R}^n}\big(\sum_{k=0}^{\infty} 2^{skq}|a^{k-h}(x,D)u_k(x)|^q\big)^{\frac{p}{q}}\,dx\Big)^{\frac{1}{p}} \le c'\|u\|_{F^{s+d}_{p,q}}. \tag{7.15}$$

In the contribution $a^{(3)}(x,D)u = \sum_{j=h}^{\infty} a_j(x,D)u^{j-h}$ one has, cf (5.3),

$$|a_j(x,D)u^{j-h}(x)| \le \sum_{k=0}^{j-h}|a_j(x,D)u_k(x)| \le \sum_{k=0}^{j} F_{a_j}(N,R2^k;x)u_k^*(N,R2^k;x). \tag{7.16}$$

Here Corollary 3.4 gives the estimate $F_{a_j} \le c_M p(a)2^{-jM}(R2^k)^{d+M}$ for $k \ge 1$, but $k=0$ can be incorporated by increasing c_M by a power of R. The sum over k can be treated by the well-known elementary inequality $\sum_{j=0}^{\infty} 2^{sjq}(\sum_{k=0}^{j}|b_k|)^q \le c\sum_{j=0}^{\infty} 2^{sjq}|b_j|^q$, valid for all $b_j \in \mathbb{C}$ and $0 < q \le \infty$ provided $s < 0$; cf [Yam86]. For $M > s$ this gives

$$\begin{aligned}\sum_{j=0}^{\infty} 2^{sjq}|a_j(x,D)u^{j-h}(x)|^q &\le \sum_{j=0}^{\infty} 2^{(s-M)jq}\Big(\sum_{k=0}^{j} c_M p(a)(R2^k)^{d+M}u_k^*(N,R2^k;x)\Big)^q\\ &\le cp(a)^q\sum_{j=0}^{\infty} 2^{(s+d)jq}u_j^*(N,R2^j;x)^q.\end{aligned} \tag{7.17}$$

By integration this clearly leads to

$$\Big(\int_{\mathbb{R}^n}\big(\sum_{j=0}^{\infty} 2^{sjq}|a_j(x,D)u^{j-h}(x)|^q\big)^{\frac{p}{q}}\,dx\Big)^{\frac{1}{p}} \le c_3 p(a)\Big\|\big(\sum_{j=0}^{\infty} 2^{(s+d)jq}u_j^*(x)^q\big)^{\frac{1}{q}}\Big\|_p. \tag{7.18}$$

Repeating the argument for (7.15) this gives $\|a^{(3)}(x,D)u\|_{F^s_{p,q}} \le c\|u\|_{F^{s+d}_{p,q}}$.

In estimates of $a^{(2)}(x,D)u$ the terms $(a^k - a^{k-h})(x,D)u_k$ can be treated similarly to those of $a^{(1)}(x,D)$; then $\|\mathscr{F}^{-1}(\psi - \psi(2^h\cdot))\|_1$ enters the constant instead of $\|\mathscr{F}^{-1}\psi\|_1$. Moreover, $a_k(x,D)(u^{k-1} - u^{k-h}) = \sum_{l=1}^{h-1} a_k(x,D)u_{k-l}$ where each term is estimated by $u_{k-l}^*(N,R2^{k-l};x)$, analogously to $a^{(1)}(x,D)u$; but the symbol factor $F_{a^k}(N,R2^{k-l};x)$ is now $\mathscr{O}(2^{(k-l)d})$, which contributes to the constant by an extra factor of the form $(\sum_{l=1}^{h-1} 2^{slq})^{1/q}$. Altogether one has

$$\begin{aligned}\Big(\int_{\mathbb{R}^n}\big(\sum_{k=0}^{\infty} 2^{skq}|(a^k - a^{k-h})(x,D)u_k(x) + a_k(x,D)(u^{k-1} - u^{k-h})|^q\big)^{\frac{p}{q}}\,dx\Big)^{\frac{1}{p}}&\\ \le c_2' p(a)\Big\|\big(\sum_{k=0}^{\infty} 2^{(s+d)kq}u_k^*(x)^q\big)^{\frac{1}{q}}\Big\|_p.&\end{aligned} \tag{7.19}$$

In case (2.25) holds, the last part of Proposition 5.3 and (7.8) show that the argument for (7.15) applies mutatis mutandis. This gives $\|a^{(2)}(x,D)u\|_{F^s_{p,q}} \le c\|u\|_{F^{s+d}_{p,q}}$, so for all $s \in \mathbb{R}$,

$$\|a_\psi(x,D)u\|_{F^s_{p,q}} \le \sum_{j=1,2,3}\|a^{(j)}(x,D)u\|_{F^s_{p,q}} \le cp(a)\|u\|_{F^{s+d}_{p,q}}. \tag{7.20}$$

Otherwise the spectra are by Proposition 5.3 only contained in balls, but the condition $s > \max(0, \frac{n}{p} - n)$ and those on r imply that $\|a^{(2)}(x,D)u\|_{F^s_{p,r}} \leq c\|u\|_{F^{s+d}_{p,q}}$; cf Lemma 7.2. This gives the above inequality with q replaced by r on the left-hand side.

Thus $a_\psi(x,D)\colon F^{s+d}_{p,q} \to F^s_{p,r}$ is continuous. Since $\mathscr{S}$ is dense in $F^s_{p,q}$ for $q < \infty$ (and $F^s_{p,\infty} \hookrightarrow F^{s'}_{p,1}$ for $s' < s$), there is no dependence on ψ. Hence $u \in D(a(x,D))$ and the above inequalities hold for $a(x,D)u$. This proves (7.11) in all cases.

The Besov case is analogous; one can interchange the order of L_p and ℓ_q and refer to the maximal inequality for scalar functions: Lemma 7.2 carries over to $B^s_{p,q}$ in a natural way for $0 < p \leq \infty$ with $r = q$ in all cases; this is well known, cf [Yam86, Joh05, JS08]. (One may also obtain (7.12) by real interpolation of (7.11), cf [Tri83, 2.4.2], but only for $0 < p < \infty$.) $\square$

The borderline analysis in (7.3) is a little simpler than the above. In fact, the proof in [Joh04, Joh05] applies to the definition by vanishing frequency modulation with the addendum that the right-hand side of (5.7) does not depend on ψ for $u \in F^d_{p,1}$, because $\mathscr{S}$ is dense there.

By duality, Theorem 7.4 extends to operators that merely fulfil the twisted diagonal condition of arbitrary real order.

Theorem 7.5. *Let $a(x,\eta)$ belong to the self-adjoint subclass $\tilde{S}^d_{1,1}(\mathbb{R}^n \times \mathbb{R}^n)$, characterised in Theorem 4.3. Then $a(x,D)$ is a bounded map for all $s \in \mathbb{R}$,*

$$a(x,D)\colon F^{s+d}_{p,q}(\mathbb{R}^n) \to F^s_{p,q}(\mathbb{R}^n), \quad 1 < p < \infty,\ 1 < q \leq \infty, \tag{7.21}$$

$$a(x,D)\colon B^{s+d}_{p,q}(\mathbb{R}^n) \to B^s_{p,q}(\mathbb{R}^n), \quad 1 < p \leq \infty,\ 1 < q \leq \infty. \tag{7.22}$$

Proof. When $p' + p = p'p$ and $q' + q = q'q$, then $F^s_{p,q}$ is the dual of $F^{-s}_{p',q'}$ since $1 < p' < \infty$ and $1 \leq q' < \infty$; cf [Tri83, 2.11], the case $q' = 1$ is covered by eg [FJ90, Rem. 5.14]. The adjoint symbol $a^*(x,\eta)$ is in $S^d_{1,1}$ by assumption, so

$$a^*(x,D)\colon F^{-s}_{p',q'}(\mathbb{R}^n) \to F^{-s-d}_{p',q'}(\mathbb{R}^n) \tag{7.23}$$

is continuous whenever $-s-d > \max(0, \frac{n}{p'} - n) = 0$, ie for $s < -d$; this follows from Theorem 7.4 since $p' \geq 1$ and $q' \geq 1$. The adjoint $a^*(x,D)^*$ is therefore bounded $F^{s+d}_{p,q} \to F^s_{p,q}$, and it equals $a(x,D)$ according to Theorem 4.6. When $s > 0$ then (7.21) also holds by Theorem 7.4.

If $d \geq 0$ the gap with $s \in [-d, 0]$ can be closed since $a(x,D) = b(x,D)\Lambda^t$ by Proposition 2.2 holds with $\Lambda^t = \operatorname{OP}((1+|\eta|^2)^{t/2})$, $t \in \mathbb{R}$ and $b(x,\eta) = a(x,\eta)(1+|\eta|^2)^{-t/2}$. The latter is of order -1 for $t = d+1$, which gives (7.21) for all s.

For the $B^s_{p,q}$ scale similar arguments apply, also for $p = \infty$. $\square$

In case $p = 2 = q$, Hörmander obtained boundedness $\big\|a(x,D)u\big\|_{H^s} \leq c\big\|u\big\|_{H^{s+d}}$ for Schwartz functions u and all $s \in \mathbb{R}$ when $a \in \tilde{S}^d_{1,1}$. This was an immediate consequence of [Hör89, Thm. 4.1], but first formulated in [Hör97, Thm. 9.4.2]. Obviously Theorem 7.5 gives a natural generalisation to the L_p-setting that relies on the definition of type $1,1$-operators.

Specialisation of Theorems 7.4–7.5 to Sobolev and Hölder–Zygmund spaces, cf (7.1)–(7.2), gives

Corollary 7.6. *Every* $a(x,D) \in \mathrm{OP}(S^d_{1,1}(\mathbb{R}^n \times \mathbb{R}^n))$ *is bounded*

$$a(x,D)\colon H^{s+d}_p(\mathbb{R}^n) \to H^s_p(\mathbb{R}^n), \quad s>0,\ 1<p<\infty, \tag{7.24}$$

$$a(x,D)\colon C^{s+d}_*(\mathbb{R}^n) \to C^s_*(\mathbb{R}^n), \quad s>0. \tag{7.25}$$

This extends to all real s *whenever* $a(x,\eta)$ *belongs to the self-adjoint subclass* $\tilde{S}^d_{1,1}(\mathbb{R}^n \times \mathbb{R}^n)$.

7.2. **Direct estimates for the self-adjoint subclass.** To complement Theorem 7.5 with similar results valid for p, q in $]0,1]$ one can exploit the paradifferential decomposition (5.7) and the pointwise estimates used above.

However, in the results below there will be an arbitrarily small loss of smoothness. The reason is that the estimates of $a^{(2)}_\psi(x,D)$ are based on a corona condition which is *asymmetric* in the sense that the outer radii grow faster than the inner ones. That is, the last part of Lemma 7.2 will now be extended to series $\sum u_j$ fulfilling the more general condition, where $0<\theta\le 1$ and $A>1$,

$$\begin{aligned} &\operatorname{supp}\mathscr{F}u_0 \subset \{\, \xi \mid |\xi| \le A2^j \,\}, \quad \text{for} \quad j\ge 0,\\ &\operatorname{supp}\mathscr{F}u_j \subset \{\, \xi \mid \tfrac{1}{A}2^{\theta j} \le |\xi| \le A2^j \,\} \quad \text{for } j \ge J \ge 1. \end{aligned} \tag{7.26}$$

This situation is probably known to experts in function spaces, but in lack of a reference it is analysed here. The techniques should be standard, so the explanations will be brief.

The main point of (7.26) is that $\sum u_j$ still converges for $s\le 0$, albeit with a loss of smoothness; cf the cases below with $s'<s$. Actually the loss is proportional to $(1-\theta)/\theta$, hence tends to ∞ for $\theta\to 0$, which reflects that convergence in some cases fails for $\theta = 0$ (take $\hat{u}_j = \frac{1}{j}\psi \in C^\infty_0$, $s=0$, $1<q\le\infty$).

Proposition 7.7. *Let* $s\in\mathbb{R}$, $0<p<\infty$, $0<q\le\infty$, $J\in\mathbb{N}$ *and* $0<\theta\le 1$ *be given; with* $q>n/(n+s)$ *if* $s>0$. *For each sequence* $(u_j)_{j\in\mathbb{N}_0}$ *in* $\mathscr{S}'(\mathbb{R}^n)$ *fulfilling the corona condition* (7.26) *together with the bound (usual modification for* $q=\infty$*)*

$$F := \big\|(\sum_{j=0}^\infty |2^{sj}u_j(\cdot)|^q)^{\frac{1}{q}}\big\|_{L_p} < \infty, \tag{7.27}$$

the series $\sum_{j=0}^\infty u_j$ *converges in* $\mathscr{S}'(\mathbb{R}^n)$ *to some* $u\in F^{s'}_{p,q}(\mathbb{R}^n)$ *with*

$$\|u\|_{F^{s'}_{p,q}} \le cF, \tag{7.28}$$

whereby the constant c *also depends on* s', *which one can take as* $s'=s$ *for* $\theta=1$, *or in case* $0<\theta<1$, *take to fulfil*

$$s'=s \quad \text{for} \quad s>\max(0,\tfrac{n}{p}-n), \tag{7.29}$$

$$s'<s/\theta \quad \text{for} \quad s\le 0,\ p\ge 1,\ q\ge 1, \tag{7.30}$$

or in general

$$s' < s - \tfrac{1-\theta}{\theta}(\max(0,\tfrac{n}{p}-n)-s)_+. \tag{7.31}$$

(Here $s'=s$ *is possible by* (7.29) *if the positive part* $(\dots)_+$ *has strictly negative argument.)*

The conclusions carry over to $B^{s'}_{p,q}$ *for any* $q\in]0,\infty]$ *when* $B := (\sum_{j=0}^\infty 2^{sjq}\|u_j\|_p^q)^{\frac{1}{q}} < \infty$.

Remark 7.8. The above restriction $q > n/(n+s)$ for $s > 0$ is not severe, for if (7.27) holds for a sum-exponent in $]0, n/(n+s)]$, then the constant F is also finite for any $q > n/(n+s)$, which yields the convergence and an estimate in a slightly larger space; cf the r in Lemma 7.2

Proof. Increasing $A \geq 1$, as we may, gives a reduction to the case $J = 1$: $u = \sum u_j$ has the contributions $0 + \dots + 0 + u_J + u_{J+1} + \dots$ and $(u_0 + \dots + u_{J-1}) + 0 + \dots$, where the former fulfils the conditions for $J = 1$; the latter trivially converges, it fulfils (7.26) for $J = 1$ if A is replaced by $A2^J$ and (7.27) as $\|u_0 + \dots + u_{J-1}\|_p \leq J2^{|s|J} F < \infty$. Hence $\|u\|_{F^{s'}_{p,q}} \leq C(c + J2^{|s|J})F$ if C is the constant from the quasi-triangle inequality.

It is first assumed that $u = \sum u_k$ converges in $\mathscr{S}'$. Then each term $\Phi_j(D)\sum u_k$ in the expression for $\|u\|_{F^{s'}_{p,q}}$ is defined; cf (7.5). Writing now $\Phi_j(\eta)$ as $\Phi(2^{-j}\eta)$ for clarity, one has

$$\Phi(2^{-j}D)\sum_{k\geq 0} u_k = \sum_{j-h\leq k\leq [j/\theta]+h} \Phi(2^{-j}D)u_k. \tag{7.32}$$

In fact, (7.26) gives an $h \in \mathbb{N}$ such that $\Phi(2^{-j}D)\mathscr{F}u_k = 0$ for all $k \notin [j-h, \frac{j}{\theta}+h]$.

To proceed it is convenient to use Marschall's inequality; cf Remark 7.3. This gives

$$|\Phi(2^{-j}D)u_k(x)| \leq c\big\|\Phi(R2^{\nu-j}\cdot)\big\|_{\dot{B}^{\frac{n}{t}}_{1,t}} M_t u_k(x), \quad \text{for} \quad 0 < t \leq 1, \tag{7.33}$$

whereby ν should be taken so large that $B(0, R2^\nu)$ contains the supports of $\Phi(2^{-j}\cdot)$ and $\hat{u}_k$; also $R \geq A$ can be arranged. Note that by Remark 7.3,

$$\big\|\Phi(R2^{\nu-j}\cdot)\big\|_{\dot{B}^{\frac{n}{t}}_{1,t}} = 2^{(\nu-j)(\frac{n}{t}-n)}\big\|\Phi(R\cdot)\big\|_{\dot{B}^{\frac{n}{t}}_{1,t}}. \tag{7.34}$$

This is applied in the following for some $t \in]0,1]$ that also fulfils $t < \min(p,q)$, and the main point is to show that, with s' as in the statement, it holds in all cases that

$$\Big(\sum_{j=0}^{\infty} 2^{s'jq}|\Phi(2^{-j}D)\sum_{k\geq 0} u_k(x)|^q\Big)^{1/q} \leq c\Big(\sum_{k=0}^{\infty} 2^{skq} M_t u_k(x)^q\Big)^{1/q}. \tag{7.35}$$

The easiest case is for $0 < q \leq 1$. As $\ell_q \hookrightarrow \ell_1$ for such q, one has

$$\begin{aligned}\sum_{j=0}^{\infty} 2^{s'jq}|\Phi(2^{-j}D)\sum_{k\geq 0} u_k(x)|^q &\leq \sum_{j=0}^{\infty}\sum_{j-h\leq k\leq j/\theta+h} 2^{s'jq}|\Phi(2^{-j}D)u_k(x)|^q \\ &\leq c\sum_{k=0}^{\infty}\sum_{\theta k-h\leq j\leq k+h} 2^{s'jq}\big\|\Phi(R2^{\nu-j}\cdot)\big\|^q_{\dot{B}^{\frac{n}{t}}_{1,t}} M_t u_k(x)^q.\end{aligned} \tag{7.36}$$

Here $\nu = j$ gives a constant for $j \geq k$, so the above is both for $s' \gtreqless 0$ estimated by

$$c\sum_{k=0}^{\infty}(h2^{s'kq} + \sum_{\theta k-h\leq j\leq k} 2^{s'jq+(\frac{n}{t}-n)(k-j)q})M_t u_k(x)^q. \tag{7.37}$$

For $\theta = 1$ the sum over j has a fixed number of terms, hence is $\mathscr{O}(2^{skq})$ for $s' = s$; cf (7.35).

In the case in (7.29) one may as $q > n/(n+s)$ arrange that $s' = s > \frac{n}{t} - n > \max(0, \frac{n}{p} - n, \frac{n}{q} - n)$ by taking t sufficiently close to $\min(p,q)$. Then the geometric series above is estimated by the last term, hence is $\mathscr{O}(2^{skq})$, as required in (7.35).

What remains of (7.31) are the cases in which $s \le \max(0, \frac{n}{p} - n)$, that is

$$s' < s \le \max(0, \tfrac{n}{p} - n, \tfrac{n}{q} - n) < \tfrac{n}{t} - n, \qquad t \in]0, \min(p,q)[. \tag{7.38}$$

By (7.31) a suitably small $t > 0$ yields $s = \theta s' + (1-\theta)(\frac{n}{t} - n)$, and since $s' - (\frac{n}{t} - n) < 0$ in the above sum an estimate by the first term gives $\mathscr{O}(2^{(s'\theta + (1-\theta)(\frac{n}{t}-n))kq}) = \mathscr{O}(2^{skq})$.

For $1 < q < \infty$ the inequality (7.35) follows by use of Hölder's inequality in (7.32), for if $q + q' = q'q$, one can for $s' < 0$ use $2^{\theta s'(k-j)}$ as a summation factor to get

$$|\Phi(2^{-j}D) \sum_{k \ge 0} u_k(x)|^q \le c \sum_{k=j-h}^{[j/\theta]+h} 2^{(k-j)s'\theta q} \|\Phi(R2^{\nu - j}\cdot)\|^q_{\dot{B}^{\frac{n}{t}}_{1,t}} M_t u_k(x)^q \Big(\frac{2^{-(\frac{1}{\theta}-1)js'\theta q'}}{2^{-s'\theta q'} - 1}\Big)^{\frac{q}{q'}}. \tag{7.39}$$

Therefore the above procedure yields an estimate of $\sum_{j=0}^{\infty} 2^{s'jq} |\Phi(2^{-j}D) \sum_{k \ge 0} u_k(x)|^q$ by

$$\sum_{k=0}^{\infty} 2^{ks'\theta q} M_t u_k(x)^q \Big(h + \sum_{\theta k - h \le j < k} 2^{(k-j)(\frac{n}{t} - n)q}\Big) \le c \sum_{k=0}^{\infty} 2^{(s'\theta + (1-\theta)(\frac{n}{t} - n))kq} M_t u_k(x)^q, \tag{7.40}$$

which again gives (7.35) by using (7.31) to arrange $s \ge s'\theta + (1-\theta)(\frac{n}{t} - n)$ for a $t \in]0,1[$. By making the last inequality strict for a slightly larger t, the argument is seen to extend to cases with $0 \le s' < s \le \max(0, \frac{n}{p} - n)$ by using $s' - (\frac{n}{t} - n) < 0$ instead of s' in Hölder's inequality. In fact, one gets $\sum 2^{(s'\theta + (1-\theta)(\frac{n}{t} - n))kq} (h 2^{h(\frac{n}{t} - n)} + (1 + h + k(1-\theta))) M_t u_k(x)^q$, which again is $\mathscr{O}(2^{skq})$ as the term $k(1-\theta)$ is harmless by the choice of t (or for $\theta = 1$). Hence (7.35) holds.

In case $s' = s > 0$, cf (7.29), one may take $s - \frac{n}{t} + n > 0$ (as for $q \le 1$) now with $2^{(k-j)(s - \frac{n}{t} + n)/2}$ as a summation factor: then $(\dots)^{q/q'} = \mathscr{O}(1)$, so the factor in front of $M_t u_k^q$ becomes

$$\sum_{\theta k - h \le j \le k + h} 2^{sjq + (k-j)(s - \frac{n}{t} + n)q/2 + (k-j)_+ (\frac{n}{t} - n)q} = \mathscr{O}(2^{skq}). \tag{7.41}$$

For $q = \infty$ a direct argument yields sup-norms weighted by $2^{s'j}$ and 2^{sk} in (7.35).

By the choice of t, the Fefferman–Stein inequality applies to (7.35), cf (7.8), whence

$$\Big(\int_{\mathbb{R}^n} \Big(\sum_{j=0}^{\infty} 2^{s'jq} |\Phi_j(D) \sum_{k \ge 0} u_k(x)|^q\Big)^{p/q} dx\Big)^{1/p} \le c\Big(\int \|2^{sk} u_k(\cdot)\|^p_{\ell_q} \, dx\Big)^{1/p} = cF. \tag{7.42}$$

Convergence is trivial for the partial sums $u^{(m)} = \sum_{j \le m} u_j$, hence for $u^{(m+M)} - u^{(m)}$. So (7.42) applies to $(0, \dots 0, u_{m+1}, \dots, u_{m+M}, 0, \dots)$, which for $q < \infty$ by majorisation for $m \to \infty$ yields

$$\|u^{(m+M)} - u^{(m)}\|_{F^{s'}_{p,q}} \le c\Big(\int_{\mathbb{R}^n} \Big(\sum_{k=m}^{\infty} 2^{skq} |u_k(x)|^q\Big)^{p/q} dx\Big)^{1/p} \searrow 0. \tag{7.43}$$

As $F^{s'}_{p,q}$ is complete, $\sum u_j$ converges to an element $u(x)$ with norm $\le cF$ according to (7.42). For $q = \infty$ there is convergence in the larger space $F^{s'-1/\theta}_{p,1}$ since the constant F remains finite if s, ∞ are replaced by $s-1$, 1; and again $\|u\|_{F^{s'}_{p,q}} \le cF$ holds by (7.42).

For the Besov case the arguments are analogous. First of all the absolute value should be replaced by the norm of L_p in (7.36), that now pertains to $0<q\le\min(1,p)$. Hölder's inequality applies in this case if $1/q+1/q'=1/\min(1,p)$; and (7.42) can be replaced by boundedness of M_t in L_p for $t<p$. Convergence is similarly shown. □

Thus prepared, one arrives at a general result for $0<p\le 1$.

Theorem 7.9. *If $a(x,\eta)$ belongs to the self-adjoint subclass $\tilde{S}^d_{1,1}(\mathbb{R}^n\times\mathbb{R}^n)$, the operator $a(x,D)$ is bounded for $0<p\le1$, $0<q\le\infty$,*

$$a(x,D)\colon F^{s+d}_{p,q}(\mathbb{R}^n)\to F^{s'}_{p,q}(\mathbb{R}^n)\quad \textit{for } s'<s\le \tfrac{n}{p}-n, \tag{7.44}$$

$$a(x,D)\colon B^{s+d}_{p,q}(\mathbb{R}^n)\to B^{s'}_{p,q}(\mathbb{R}^n)\quad \textit{for } s'<s\le \tfrac{n}{p}-n. \tag{7.45}$$

Proof. The theorem follows by elaboration of the proof of Theorem 6.7. By applying the last part of Theorem 7.4 to the difference $a-a_{\chi,1}$, the question is again reduced to the case in which $\hat{a}(\xi,\eta)\neq0$ only holds for $\max(1,|\xi+\eta|)\le|\eta|$; cf (6.8).

Under this assumption, $a^{(1)}(x,D)u$ and $a^{(3)}(x,D)u$ are for all $s\in\mathbb{R}$ covered by Theorem 7.4; cf (7.20). Thus it suffices to estimate the series in (6.9) for fixed $s'<s\le\frac{n}{p}-n$; notice that simple embeddings and Remark 7.8 gives a reduction to the case $q>n/(n+s)$ if $s>0$.

Now $\theta\in]0,1[$ can be taken so small that $s'<s-\frac{\theta}{1-\theta}(\frac{n}{p}-n-s)$, which is the last condition in Proposition 7.7 with $1-\theta$ instead of θ. Then $\varepsilon=2^{-k\theta}$ in (6.14) clearly gives

$$2^{k(s+M)}|a_{k,\chi,\varepsilon}(x,D)v_k(x)|\le cv_k^*(N,R2^k;x)2^{k(s+d)}2^{-k\theta(\sigma-1-N-M/\theta)}. \tag{7.46}$$

Here one may first of all take $N>n/\min(p,q)$ so that (7.8) applies. Secondly, since by assumption $a(x,\eta)$ fulfils the twisted diagonal condition (2.31) of any real order, σ can for any M (with θ fixed as above) be chosen so that $2^{-k\theta(\sigma-1-N-M/\theta)}\le1$. This gives

$$\begin{aligned}(\int\|2^{k(s+M)}a_{k,\chi,\varepsilon}(x,D)v_k(\cdot)\|^p_{\ell_q}\,dx)^{\frac1p}&\le c(\int\|2^{k(s+d)}v_k^*(N,R2^k;\cdot)\|^p_{\ell_q}\,dx)^{\frac1p}\\&\le c'(\int\|2^{k(s+d)}v_k(\cdot)\|^p_{\ell_q}\,dx)^{\frac1p}\le c''\|u\|_{F^{s+d}_{p,q}}.\end{aligned} \tag{7.47}$$

Here the last inequality follows from the (quasi-)triangle inequality in ℓ_q and L_p.

Since the spectral support rule and Proposition 5.3 imply that $a_{k,\chi,\varepsilon}(x,D)v_k$ also has its spectrum in $B(0,2R2^k)$, the above estimate allows application of Lemma 7.2, if M is so large that

$$M>0,\quad M+s>0,\quad M+s>\tfrac{n}{p}-n. \tag{7.48}$$

This gives convergence of $\sum a_{k,\chi,2^{-k\theta}}(x,D)v_k$ to a function in $F^{s+M}_{p,\infty}$ fulfilling

$$\|\sum_{k=1}^{\infty}a_{k,\chi,2^{-k\theta}}(x,D)v_k\|_{F^{s+M}_{p,\infty}}\le c\|u\|_{F^{s+d}_{p,q}}. \tag{7.49}$$

On the left-hand side the embedding $F^{s+M}_{p,\infty}\hookrightarrow F^{s}_{p,q}$ applies, of course.

For the remainder $\sum_{k=1}^{\infty} b_k(x,D)v_k$, cf (6.10) ff, note that (7.47) holds for $M=0$ with the same σ. If combined with (7.19), it follows by the (quasi-)triangle inequality that

$$\int \big\|2^{ks} b_k(x,D)v_k(\cdot)\big\|_{\ell_q}^p \, dx \le \int \big\|2^{ks}(a_k(x,D) - a_{k,\chi,2^{-k\theta}}(x,D))v_k(\cdot)\big\|_{\ell_q}^p \, dx \le c\|u\|_{F^{s+d}_{p,q}}^p. \quad (7.50)$$

In addition the series was previously shown to fulfil a corona condition with inner radius $2^{(1-\theta)k}$ for all large k, cf (6.19), so Proposition 7.7 applies. By the choice of θ, this gives

$$\big\|\sum_{k=1}^{\infty} b_k(x,D)v_k\big\|_{F^{s'}_{p,q}} \le c\|u\|_{F^{s+d}_{p,q}}. \quad (7.51)$$

In $a_\psi^{(2)}(x,D)u$ the other contribution $\sum(a^k(x,D) - a^{k-h}(x,D))u_k$, cf (6.9), can be treated similarly. This was also done in the proof of Theorem 6.7, where in particular (6.14) was shown to hold for $(a^k - a^{k-h})_{\chi,\varepsilon}(x,D)u_k$, with just a change of the constant. Consequently (7.46) carries over, and with (7.48) the same arguments as for (7.49), (7.51) give

$$\big\|\sum_{k=h}^{\infty}(a^k - a^{k-h})_{\chi,\varepsilon}(x,D)u_k\big\|_{F^{s+M}_{p,\infty}} + \big\|\sum_{k=h}^{\infty}\tilde{b}_k(x,D)u_k\big\|_{F^{s'}_{p,q}} \le c\|u\|_{F^{s+d}_{p,q}}. \quad (7.52)$$

Altogether the estimates (7.49), (7.51), (7.52) show that

$$\big\|a_\psi^{(2)}(x,D)u\big\|_{F^{s'}_{p,q}} \le c\big\|u\big\|_{F^{s+d}_{p,q}}. \quad (7.53)$$

Via the decomposition (5.7), $a_\psi(x,D)$ is therefore a bounded linear map $F^{s+d}_{p,q} \to F^{s'}_{p,q}$. Since $\mathscr{S}$ is dense for $q<\infty$ (a case one can reduce to), there is no dependence on the modulation function ψ, so the type $1,1$-operator $a(x,D)$ is defined and continuous on $F^{s+d}_{p,q}$ as stated.

The arguments are similar for the Besov spaces: it suffices to interchange the order of the norms in ℓ_q and L_p, and to use the estimate in (7.8) for each single k. $\square$

The proof extends to cases with $0<p\le\infty$ when $s'<s\le\max(0,\frac{n}{p}-n)$, but this barely fails to reprove Theorem 7.5, so only $p\le 1$ is included in Theorem 7.9.

When taken together, Theorems 7.4, 7.5 and 7.9 give a satisfactory L_p-theory of operators $a(x,D)$ in $\mathrm{OP}(\tilde{S}^d_{1,1})$, inasmuch as for the domain $D(a(x,D))$ they cover all possible s, p. Only a few of the codomains seem barely unoptimal, and these all concern cases with $0<q<1$ or $0<p\le 1$; cf the parameters r in Theorem 7.4 and s' in Theorem 7.9.

One particular interest of Theorem 7.9 is that $F^0_{p,2}(\mathbb{R}^n)$ in addition identifies with the so-called local Hardy space $h_p(\mathbb{R}^n)$ for $0<p\le 1$; cf [Tri83] and especially [Tri92, Ch. 1.4]. In this case Theorem 7.9 gives boundedness $a(x,D)\colon h_p(\mathbb{R}^n) \to F^{s'}_{p,2}(\mathbb{R}^n)$ for every $s'<0$, but this can probably be improved in view of recent results:

Remark 7.10. Extensions to $h_p(\mathbb{R}^n)$ of operators in the self-adjoint subclass $\mathrm{OP}(\tilde{S}^0_{1,1})$ were treated by Hounie and dos Santos Kapp [HdSK09], who used atomic estimates to carry over the L_2-boundedness of Hörmander [Hör89, Hör97] to h_p, ie to obtain estimates with $s'=s=0$. However, they worked without a precise definition of type $1,1$-operators. Torres [Tor90] obtained extensions by continuity using the atomic decompositions in [FJ90], but for $s<0$ he relied on conditions on the adjoint $a(x,D)^*$ rather than on the symbol $a(x,\eta)$ itself. In the $F^s_{p,q}$-scales, general type $1,1$-operators were first estimated by Runst [Run85], though with insufficient control of the spectra as noted in [Joh05] (a remedy is provided by Appendix B).

Remark 7.11. In addition to Theorem 7.9, its proof gives that when $a(x,D)$ fulfils the twisted diagonal condition of a specific order $\sigma>0$, then for $1\le p\le\infty$

$$B^s_{p,q}\cup F^s_{p,q}\subset D(a(x,D)) \quad \text{for} \quad s>-\sigma+[n/2]+2. \tag{7.54}$$

While this does provide a result in the L_p set-up, it is hardly optimal; cf Hörmander's condition $s>-\sigma$ for $p=2$, recalled in (2.34).

8. Final remarks

In view of the satisfying results on type $1,1$-operators in $\mathscr{S}'(\mathbb{R}^n)$, cf Section 6, and the continuity results in the scales H^s_p, C^s_*, $F^s_{p,q}$ and $B^s_{p,q}$ presented in Section 7, their somewhat unusual definition by vanishing frequency modulation in Definition 2.1 should be well motivated.

As an open problem, it remains to characterise the type $1,1$-operators $a(x,D)$ that are everywhere defined and continuous on $\mathscr{S}'(\mathbb{R}^n)$. For this it was shown above to be sufficient that $a(x,\eta)$ is in $\tilde{S}^d_{1,1}(\mathbb{R}^n\times\mathbb{R}^n)$, and it could of course be conjectured that this is necessary as well.

Similarly, since the works of Bourdaud and Hörmander, cf [Bou83, Ch. IV], [Bou88a], [Hör88, Hör89] and also [Hör97], it has remained an open problem to determine

$$\mathbb{B}(L_2(\mathbb{R}^n))\cap \mathrm{OP}(S^0_{1,1}). \tag{8.1}$$

Indeed, this set was shown by Bourdaud to contain the self-adjoint subclass $\mathrm{OP}(\tilde{S}^0_{1,1})$, and this sufficient condition led some authors to the misleading statement that eg lack of L_2-boundedness for $\mathrm{OP}(S^0_{1,1})$ is "attributable to the lack of self adjointness". But self-adjointness is not necessary, since already Bourdaud, by modification of Ching's operator (2.9), gave an example [Bou88a, p. 1069] of an operator $\sigma(x,D)$ in $\mathbb{B}(L_2)\bigcap \mathrm{OP}(S^0_{1,1}\setminus\tilde{S}^0_{1,1})$; ie $\sigma(x,D)^*$ is not of type $1,1$.

However, it could be observed that $N_{\chi,\varepsilon,\alpha}(a_\theta)=\mathscr{O}(\varepsilon^{n/2-|\alpha|})$ for Ching's symbol a_θ by Lemma 2.10, and that this is sharp for the L_2-unbounded version of $a_\theta(x,D)$ by the last part of Example 2.11. So it is natural to consider the condition for $\varepsilon\to 0$,

$$N_{\chi,\varepsilon,\alpha}(a)=o(\varepsilon^{n/2-|\alpha|}). \tag{8.2}$$

It is conjectured that this is necessary for L_2-continuity of a given $a(x,D)$ in $\mathrm{OP}(S^0_{1,1})$.

Appendix A. Dyadic corona criteria

As a general tool, convergence of a series $\sum_{j=0}^\infty u_j$ of temperate distributions follows if the u_j fulfil both a growth condition and have their spectra in suitable dyadic coronas. This follows from Lemma A.1, which for $\theta_0=\theta_1=1$ was given by Coifman and Meyer [MC97, Ch. 15] without arguments. (A proof of this case can be found in [JS08].)

The refined version in Lemma A.1 allows the inner and outer radii of the spectra to grow at different exponential rates (2^{θ_0} and 2^{θ_1}), even though the number of overlapping spectra increases with j. This is crucial here, so a full proof is given.

Lemma A.1. 1° *Let* $(u_j)_{j\in\mathbb{N}_0}$ *be a sequence in* $\mathscr{S}'(\mathbb{R}^n)$ *fulfilling that there exist* $A>1$ *and* $\theta_1>\theta_0>0$ *such that* $\operatorname{supp}\hat{u}_0 \subset \{\,\xi \mid |\xi|\le A\,\}$ *while for* $j\ge 1$

$$\operatorname{supp}\hat{u}_j \subset \{\,\xi \mid \tfrac{1}{A}2^{j\theta_0} \le |\xi| \le A2^{j\theta_1}\,\}, \tag{A.1}$$

and that for suitable constants $C\ge 0$, $N\ge 0$,

$$|u_j(x)| \le C2^{jN\theta_1}(1+|x|)^N \text{ for all } j\ge 0. \tag{A.2}$$

Then $\sum_{j=0}^\infty u_j$ *converges rapidly in* $\mathscr{S}'(\mathbb{R}^n)$ *to a distribution* u*, for which* $\hat{u}$ *is of order* N*.*

2° *For every* $u\in\mathscr{S}'(\mathbb{R}^n)$ *both* (A.1) *and* (A.2) *are fulfilled for* $\theta_0=\theta_1=1$ *by the functions* $u_0=\Phi_0(D)u$ *and* $u_j=\Phi(2^{-j}D)u$ *when* $\Phi_0,\Phi\in C_0^\infty(\mathbb{R}^n)$ *and* $0\notin\operatorname{supp}\Phi$*. In particular this is the case for a Littlewood–Paley decomposition* $1=\Phi_0+\sum_{j=1}^\infty\Phi(2^{-j}\xi)$*.*

Proof. In 2° it is clear that Φ is supported in a corona, say $\{\,\xi \mid \frac{1}{A}\le|\xi|\le A\,\}$ for a large $A>0$; hence (A.1). (A.2) follows from the proof of Lemma 3.1.

The proof of 1° exploits a well-known construction of an auxiliary function: taking $\psi_0\in C_0^\infty(\mathbb{R}^n)$ depending on $|\xi|$ alone and so that $0\le\psi\le 1$ with $\psi_0(\xi)=1$ for $|\xi|\le 1/(2A)$ while $\psi_0(\xi)=0$ for $|\xi|\ge 1/A$, then

$$\frac{d}{dt}\psi_0(\frac{\xi}{t}) = \psi(\frac{\xi}{t})\frac{1}{t} \quad \text{for} \quad \psi(\xi)=-\xi\cdot\nabla\psi_0(\xi), \tag{A.3}$$

which by integration for $1\le t\le\infty$ gives an uncountable partition of unity

$$1=\psi_0(\xi)+\int_1^\infty \psi(\frac{\xi}{t})\,\frac{dt}{t}, \quad \xi\in\mathbb{R}^n. \tag{A.4}$$

Clearly the support of $\psi(\xi/t)$ is compact and given by $A|\xi|\le t\le 2A|\xi|$ when ξ is fixed. For $j\ge 1$ this implies

$$\hat{u}_j=\hat{u}_j\psi_0+\hat{u}_j\int_1^\infty \psi(\frac{\xi}{t})\,\frac{dt}{t}=\hat{u}_j\int_{2^{j\theta_0}}^{A^2 2^{j\theta_1+1}}\psi(\frac{\xi}{t})\,\frac{dt}{t}. \tag{A.5}$$

Defining $\psi_j\in C_0^\infty(\mathbb{R}^n)$ as the last integral here, $\psi_j=1$ on $\operatorname{supp}\hat{u}_j$; so if $\varphi\in\mathscr{S}$,

$$|\langle u_j,\overline{\varphi}\rangle| \le \big\|(1+|x|^2)^{-\frac{N+n}{2}}u_j\big\|_2\big\|(1+|x|^2)^{\frac{N+n}{2}}\mathscr{F}^{-1}(\psi_j\hat{\varphi})\big\|_2. \tag{A.6}$$

The first norm is $\mathscr{O}(2^{N\theta_1 j})$ by (A.2). For the second, note that

$$\operatorname{supp}\psi_j\subset\{\,\xi\in\mathbb{R}^n \mid A^{-1}2^{j\theta_0-1}\le|\xi|\le A2^{j\theta_1+1}\,\} \tag{A.7}$$

and $\|D^\alpha\psi_j\|_\infty\le 2^{-j\theta_0|\alpha|}\|D^\alpha\psi\|_\infty/|\alpha|$ for $\alpha\ne 0$ while $\|\psi_j\|_\infty\le\operatorname{diam}(\psi(\mathbb{R}))\le 1$ by (A.3). In addition the identity $(1+|x|^2)^{N+n}\mathscr{F}^{-1}=\mathscr{F}^{-1}(1-\Delta)^{N+n}$ gives for arbitrary $k>0$,

$$\begin{aligned}&\|(1+|x|^2)^{N+n}\mathscr{F}^{-1}(\psi_j\hat{\varphi})\|_2\\&\qquad\le\sum_{|\alpha|,|\beta|\le N+n}c_{\alpha,\beta}\|D^\alpha\psi_j\|_\infty\|(1+|\xi|)^{k+n/2}D^\beta\hat{\varphi}\|_\infty\Big(\int_{2^{j\theta_0-1}/A}^\infty r^{-1-2k}\,dr\Big)^{1/2}.\end{aligned} \tag{A.8}$$

Here $\|D^\alpha\psi_j\|_\infty=\mathscr{O}(1)$, so because of the L_2-norm the above is $\mathscr{O}(2^{-jk\theta_0})$ for every $k>0$.

Hence $\langle u_j, \overline{\varphi} \rangle = \mathscr{O}(2^{j(\theta_1 N-\theta_0 k)})$, so $k > N\theta_1/\theta_0$ yields that $\sum_{j=0}^{\infty}\langle u_j, \varphi \rangle$ converges. $\square$

Remark A.2. The above proof yields that the conjunction of (A.1) and (A.2) implies $\langle u_j, \varphi \rangle = \mathscr{O}(2^{-jN})$ for arbitrary $N > 0$, hence there is *rapid* convergence of $u = \sum_{j=0}^{\infty} u_j$ in the sense that $\langle u - \sum_{j<k} u_j, \varphi \rangle = \sum_{j\geq k}\langle u_j, \varphi \rangle = \mathscr{O}(2^{-kN})$ for $N > 0$, $\varphi \in \mathscr{S}(\mathbb{R}^n)$.

APPENDIX B. THE SPECTRAL SUPPORT RULE

To control the spectrum of $x \mapsto a(x,D)u$, ie the support of $\xi \mapsto \mathscr{F}a(x,D)u$, there is a simple rule which is recalled here for the reader's convenience.

Writing $\mathscr{F}a(x,D)\mathscr{F}^{-1}(\hat{u})$ instead, it is clear the question is how the support of $\mathscr{F}u$ is changed by the conjugated operator $\mathscr{F}a(x,D)\mathscr{F}^{-1}$. Since this has $\mathscr{K}(\xi,\eta) = (2\pi)^{-n}\hat{a}(\xi-\eta,\eta)$ as its distribution kernel, cf (1.11), one should expect the spectrum of $a(x,D)u$ to be contained in

$$\Xi := \operatorname{supp}\mathscr{K} \circ \operatorname{supp}\mathscr{F}u = \{\, \xi \in \mathbb{R}^n \mid \exists \eta \in \operatorname{supp}\hat{u} \colon (\xi,\eta) \in \operatorname{supp}\mathscr{K} \,\}. \tag{B.1}$$

This is indeed the case if $\operatorname{supp}\mathscr{F}u \Subset \mathbb{R}^n$, as was proved in [Joh05], while in general one should use the closure $\overline{\Xi}$ instead, as shown in [Joh08b]:

Theorem B.1. *Let $a \in S^{\infty}_{1,1}(\mathbb{R}^n\times\mathbb{R}^n)$ and suppose $u \in D(a(x,D))$ is such that, for some $\psi \in C_0^{\infty}(\mathbb{R}^n)$ equalling* 1 *around the origin,* (2.4) *holds in the topology of $\mathscr{S}'(\mathbb{R}^n)$. Then it holds that*

$$\operatorname{supp}\mathscr{F}(a(x,D)u) \subset \overline{\Xi}, \tag{B.2}$$

$$\Xi = \big\{\, \xi+\eta \;\big|\; (\xi,\eta) \in \operatorname{supp}\mathscr{F}_{x\to\xi}a,\ \eta \in \operatorname{supp}\mathscr{F}u \,\big\}. \tag{B.3}$$

When $u \in \mathscr{F}^{-1}\mathscr{E}'(\mathbb{R}^n)$ the $\mathscr{S}'$-convergence holds automatically and Ξ is closed for such u.

The reader is referred to [Joh08b] for the deduction of this from the kernel formula. Note that it suffices to take any $v \in C_0^{\infty}(\mathbb{R}^n)$ with support disjoint from Ξ and verify that

$$\langle \mathscr{F}a(x,D)\mathscr{F}^{-1}\hat{u}, v \rangle = \langle \mathscr{K}, v\otimes\hat{u} \rangle = 0. \tag{B.4}$$

Although the expression to the right makes sense as $\langle (v\otimes\hat{u})\mathscr{K}, 1 \rangle$ (as noted in [Joh08b], using the remarks to [Hör85, Def. 3.1.1]), it is in general not trivial to justify the first equality sign.

Remark B.2. There is a simple proof of (B.2) in case $\hat{u} \in \mathscr{E}'$ and $a \in S^d_{1,0}$: When v is as above and $\operatorname{supp}\hat{u}$ is compact, (B.1) yields $\operatorname{dist}(\operatorname{supp}\mathscr{K}, \operatorname{supp}(v\otimes\hat{u})) > 0$. So with $\hat{u}_\varepsilon = \varphi_\varepsilon * \hat{u}$ for some $\varphi \in C_0^{\infty}(\mathbb{R}^n)$ with $\hat{\varphi}(0) = 1$, $\varphi_\varepsilon = \varepsilon^{-n}\varphi(\cdot/\varepsilon)$, all sufficiently small $\varepsilon > 0$ give

$$\operatorname{supp}\mathscr{K} \bigcap \operatorname{supp} v\otimes\hat{u}_\varepsilon = \emptyset. \tag{B.5}$$

Therefore one has, since $\mathscr{F}a(x,D)\mathscr{F}^{-1}$ is continuous in $\mathscr{S}'$ and $\hat{u}_\varepsilon \in C_0^{\infty}(\mathbb{R}^n)$,

$$\langle \mathscr{F}a(x,D)\mathscr{F}^{-1}\hat{u}, v \rangle = \lim_{\varepsilon\to 0}\langle \mathscr{F}a(x,D)\mathscr{F}^{-1}\hat{u}_\varepsilon, v \rangle = \lim_{\varepsilon\to 0}\langle \mathscr{K}, v\otimes\hat{u}_\varepsilon \rangle = 0. \tag{B.6}$$

The argument in Remark B.2 actually suffices for the applications of Theorem B.1 in the present paper. Indeed, it is clear from Remark 5.1 that the summands $a^{k-h}(x,D)u_k$ etc, that appear in the paradifferential decomposition (5.7), all can be rewritten in terms of symbols in $S^{-\infty}$ without changing the set Ξ.

Further comments on Theorem B.1 can be found in Remark 5.4 and the introduction.

REFERENCES

[Bon81] J.-M. Bony, *Calcul symbolique et propagations des singularités pour les équations aux dérivées partielles non linéaires*, Ann. scient. Éc. Norm. Sup. **14** (1981), 209–246.

[Bou82] G. Bourdaud, *L_p-estimates for certain non-regular pseudo-differential operators*, Comm. Partial Differential Equations **7** (1982), 1023–1033.

[Bou83] ______, *Sur les opérateurs pseudo-différentiels à coefficients peu reguliers*, Thèse, Univ. de Paris-Sud, 1983.

[Bou88a] ______, *Une algèbre maximale d'opérateurs pseudo-différentiels*, Comm. Partial Differential Equations **13** (1988), no. 9, 1059–1083.

[Bou88b] ______, *Une algèbre maximale d'opérateurs pseudo-différentiels de type* $1,1$, Séminaire 1987–1988, Équations aux Dérivées Partielles, École Polytech., Palaiseau, 1988, pp. VII1–VII17.

[Bou95] A. Boulkhemair, *Remarque sur la quantification de Weyl pour la classe de symboles $S^0_{1,1}$*, C. R. Acad. Sci. Paris Sér. I Math. **321** (1995), no. 8, 1017–1022.

[Bou99] ______, *L^2 estimates for Weyl quantization*, J. Funct. Anal. **165** (1999), no. 1, 173–204.

[Chi72] Chin Hung Ching, *Pseudo-differential operators with nonregular symbols*, J. Differential Equations **11** (1972), 436–447.

[CM78] R. R. Coifman and Y. Meyer, *Au delà des opérateurs pseudo-différentiels*, Astérisque, vol. 57, Société Mathématique de France, Paris, 1978.

[DJ84] G. David and J.-L. Journé, *A boundedness criterion for generalized Calderón-Zygmund operators*, Ann. of Math. (2) **120** (1984), 371–397.

[FJ90] M. Frazier and B. Jawerth, *A discrete transform and decomposition of distribution spaces*, J. Func. Anal. **93** (1990), 34–170.

[HdSK09] J. Hounie and R. A. dos Santos Kapp, *Pseudodifferential operators on local Hardy spaces*, J. Fourier Anal. Appl. **15** (2009), 153–178.

[Hér02] F. Hérau, *Melin inequality for paradifferential operators and applications*, Comm. Partial Differential Equations **27** (2002), 1659–1680.

[Hör85] L. Hörmander, *The analysis of linear partial differential operators*, Grundlehren der mathematischen Wissenschaften, Springer Verlag, Berlin, 1983, 1985.

[Hör88] ______, *Pseudo-differential operators of type* $1,1$, Comm. Partial Differential Equations **13** (1988), no. 9, 1085–1111.

[Hör89] ______, *Continuity of pseudo-differential operators of type* $1,1$, Comm. Partial Differential Equations **14** (1989), no. 2, 231–243.

[Hör97] ______, *Lectures on nonlinear hyperbolic differential equations*, Mathématiques & Applications, vol. 26, Springer Verlag, Berlin, 1997.

[Joh95] J. Johnsen, *Pointwise multiplication of Besov and Triebel–Lizorkin spaces*, Math. Nachr. **175** (1995), 85–133.

[Joh04] ______, *Domains of type* $1,1$ *operators: a case for Triebel–Lizorkin spaces*, C. R. Acad. Sci. Paris Sér. I Math. **339** (2004), no. 2, 115–118.

[Joh05] ______, *Domains of pseudo-differential operators: a case for the Triebel–Lizorkin spaces*, J. Function Spaces Appl. **3** (2005), 263–286.

[Joh08a] ______, *Parametrices and exact paralinearisation of semi-linear boundary problems*, Comm. Part. Diff. Eqs. **33** (2008), 1729–1787.

[Joh08b] ______, *Type* $1,1$*-operators defined by vanishing frequency modulation*, New Developments in Pseudo-Differential Operators (L. Rodino and M. W. Wong, eds.), Operator Theory: Advances and Applications, vol. 189, Birkhäuser, 2008, pp. 201–246.

[Joh10a] ______, *Pointwise estimates of pseudo-differential operators*, Tech. Report R-2010-12, Aalborg University, 2010, (preprint).

[Joh10b] ______, *Simple proofs of nowhere-differentiability for Weierstrass's function and cases of slow growth*, J. Fourier Anal. Appl. **16** (2010), 17–33.

[JS07] J. Johnsen and W. Sickel, *A direct proof of Sobolev embeddings for quasi-homogeneous Lizorkin–Triebel spaces with mixed norms*, J. Function Spaces Appl. **5** (2007), 183–198.

[JS08] J. Johnsen and W. Sickel, *On the trace problem for Lizorkin–Triebel spaces with mixed norms*, Math. Nachr. **281** (2008), 1–28.

[KgN78] H. Kumano-go and M. Nagase, *Pseudo-differential operators with non-regular symbols and applications*, Funkcial. Ekvac. **21** (1978), 151–192.

[Lan06] D. Lannes, *Sharp estimates for pseudo-differential operators with symbols of limited smoothness and commutators*, J. Funct. Anal. **232** (2006), 495–539.

[Mar85] J. Marschall, *Pseudo-differential operators with non-regular symbols*, Ph.D. thesis, Free University of Berlin, 1985.

[Mar91] ______, *Weighted parabolic Triebel spaces of product type. Fourier multipliers and pseudo-differential operators*, Forum Math. **3** (1991), no. 5, 479–511.

[MC97] Y. Meyer and R. R. Coifman, *Wavelets*, Cambridge University Press, Cambridge, 1997.

[Mey81a] Y. Meyer, *Régularité des solutions des équations aux dérivées partielles non linéaires (d'après J.-M. Bony)*, Bourbaki Seminar, Vol. 1979/80, Lecture Notes in Math., vol. 842, Springer, Berlin, 1981, pp. 293–302.

[Mey81b] ______, *Remarques sur un théorème de J.-M. Bony*, Proceedings of the Seminar on Harmonic Analysis (Pisa, 1980), 1981, pp. 1–20.

[Pee76] J. Peetre, *New thoughts on Besov spaces*, Duke Univ. Math. Series, Durham, 1976.

[PR78] C. Parenti and L. Rodino, *A pseudo differential operator which shifts the wave front set*, Proc. Amer. Math. Soc. **72** (1978), 251–257.

[RS96] T. Runst and W. Sickel, *Sobolev spaces of fractional order, Nemytskiĭ operators and non-linear partial differential equations*, Nonlinear analysis and applications, vol. 3, de Gruyter, Berlin, 1996.

[Run85] T. Runst, *Pseudodifferential operators of the "exotic" class* $L^0_{1,1}$ *in spaces of Besov and Triebel-Lizorkin type*, Ann. Global Anal. Geom. **3** (1985), no. 1, 13–28.

[Ste93] E. M. Stein, *Harmonic analysis: real-variable methods, orthogonality, and oscillatory integrals*, Princeton Mathematical Series, vol. 43, Princeton University Press, Princeton, NJ, 1993.

[Tor90] R.H. Torres, *Continuity properties of pseudodifferential operators of type* $1,1$, Comm. Partial Differential Equations **15** (1990), 1313–1328.

[Tri77] H. Triebel, *Multiplication properties of the spaces* $B^s_{p,q}$ *and* $F^s_{p,q}$, Ann. Mat. Pura Appl. **113** (1977), 33–42.

[Tri83] ______, *Theory of function spaces*, Monographs in mathematics, vol. 78, Birkhäuser Verlag, Basel, 1983.

[Tri92] ______, *Theory of function spaces II*, Monographs in mathematics, vol. 84, Birkhäuser Verlag, Basel, 1992.

[Yam86] M. Yamazaki, *A quasi-homogeneous version of paradifferential operators, I. Boundedness on spaces of Besov type*, J. Fac. Sci. Univ. Tokyo Sect. IA, Math. **33** (1986), 131–174.

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